

Boundaries of slices of quasifuchsian space

by

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Declaration

I hereby declare that the dissertation represents my own work and has not been previously submitted to this or any other institution for any degree, diploma or other qualification. Chapter 4 is based on [Goodman06], written during my period of study for this degree.

Abstract

We prove that the boundaries of the Maskit and Bers slices contain an uncountable, dense set of points about which the boundary spirals infinitely. The set of points about which we prove the boundary spirals infinitely has zero measure and is akin to a countable union of Cantor sets. On the basis of strong numerical evidence, we conjecture that in fact the boundary spirals infinitely at almost all points in the boundary. We further conjecture that the Hausdorff dimension of the Maskit slice is less than 1.25.

Chapter 1

Introduction

The purpose of this thesis is to provide a picture, partly proven and partly conjectural on the basis of strong numerical evidence, of the shape of the boundaries of various slices of quasifuchsian space. We will focus on the Maskit and Bers slices, which have particularly nice algebraic structures.

Roughly speaking, a quasifuchsian manifold is a hyperbolic 3-manifold obtained by taking a once-punctured torus crossed with an interval. This is illustrated in the leftmost picture in figure 2.2. Quasifuchsian space $\mathcal{QF}$ is, again roughly speaking, the space of hyperbolic geometries of a quasifuchsian manifold. This object is 2-complex dimensional, and is therefore quite difficult to visualise. We will work with slices of quasifuchsian space, which are 1-complex dimensional subsets, somewhat like its intersection with a plane.

The boundary of quasifuchsian space has a very detailed, seemingly fractal structure. You can see this in figure 1.1 which shows a portion of the Maskit slice of $\mathcal{QF}$. The fine detail appears at every scale. Indeed, zooming in on any area of the Maskit slice gives a picture which hardly differs from figure 1.1 at all. It has not been proven that the Maskit slice is self-similar, but it certainly seems to be. For some partial results, see [Miyachi03].

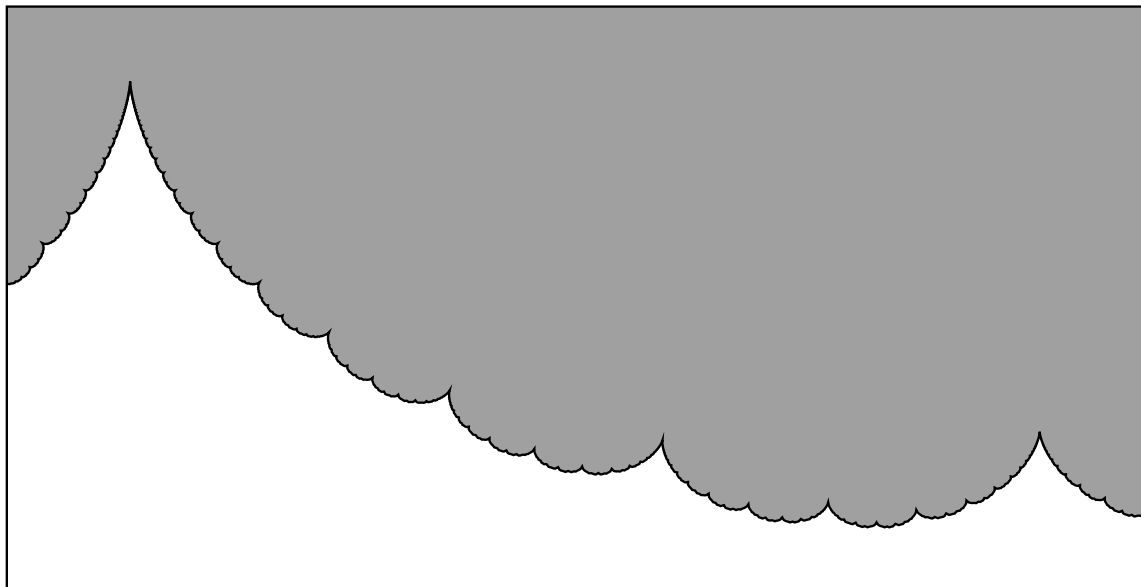


Figure 1.1: Detail from the Maskit slice, with the interior shaded grey

In a self-similar set where the similarities involve a rotation as well as a scaling, spirals appear naturally. Well known pictures of the Mandelbrot set, Julia sets and so forth vividly illustrate this (or see figure 2.1). If slices of $\mathcal{QF}$ are indeed self-similar, then we would expect to see spirals appearing in them too. On the face of it, looking at figure 1.1 doesn't immediately suggest that there are spirals in the boundary. Figure 1.2 on the other hand shows a series of zooms into the

Maskit slice, and it is apparent that the picture is slowly rotating. We imagine that this process could be continued indefinitely. The spirals are there, but they appear at such tiny scales that it is impossible to see them in any one picture. The main aim of this thesis is to show that these spirals really do exist. We prove in chapter 4 that there is a dense and uncountable set of points at which the boundary spirals infinitely. This set has zero measure – it is akin to a countable union of Cantor sets. In chapter 5 we conjecture on the basis of strong numerical evidence that in fact the boundary spirals at almost all points. Using related methods and evidence, we also make some conjectures about the Hausdorff dimension of the Maskit slice.

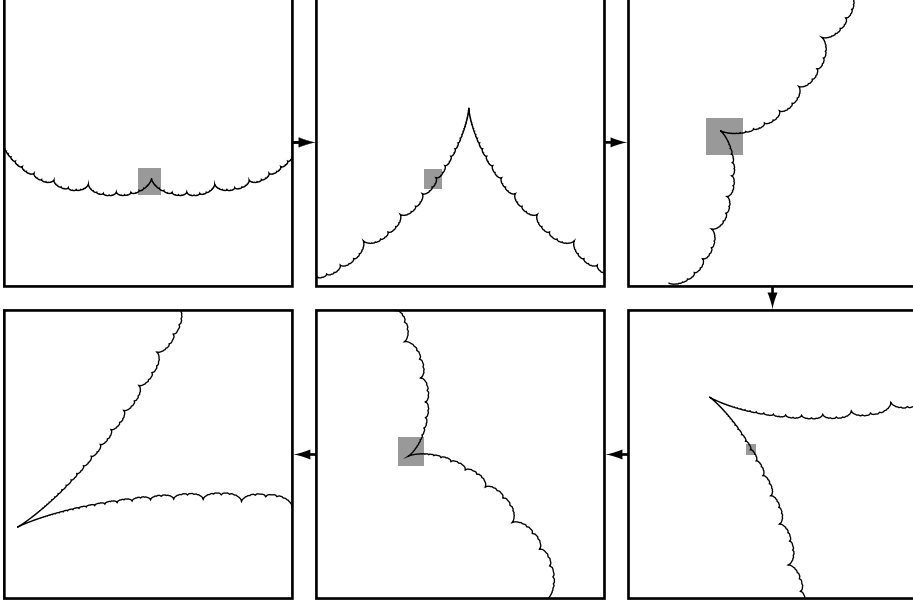


Figure 1.2: Successive zooms into the Maskit slice suggesting spiralling. The figures go from the top left to the bottom left in a clockwise order. The grey rectangles show the area being zoomed to in the next frame.

The Maskit and Bers slices embedded in $\mathbb{C}$ are simply connected sets with non-self-intersecting boundaries (see [Minsky99]). It is apparent from looking at the pictures that there is a seemingly dense set of special points, where the boundary of the slice sharply points inwards. These points are the cusp points of the slice, and they are particularly important as they give us a very good handle for studying the slice. These points are indeed dense in the boundary (see [McMullen91]). They are called cusp points because at these points there are cusps in the associated 3-manifold, but by coincidence the shape of the boundary is itself cusp shaped at the cusp points (see [Miyachi03]), which introduces an unfortunate ambiguity about the word “cusp”.

It turns out that each cusp can be labelled with a rational number p/q , and that these correctly order the cusps. We will use the notation $\mu_{p/q}$ for the location of the p/q -cusp in whatever slice we are looking at. There is also a curve associated to each cusp, called the p/q -pleating ray, written $\wp_{p/q}$. These rays are entirely contained within the slice. For the Maskit slice, they turn out to be defined by a polynomial equation associated to each cusp. Each pleating ray starts at $\mu_{p/q}$ and initially moves in the direction in which the cusp is pointing. See figure 1.3.

Figure 1.4 shows part of the Maskit slice with many pleating rays shown. As you can see, they have a very interesting structure. The approach we take in this thesis is to show first that there is an indefinitely large amount of spiralling at the cusp points in the boundary. Roughly speaking, if we assigned to each cusp point $\mu_{p/q}$ an integer $S_{p/q}$ defined to be the number of times the pleating ray $\wp_{p/q}$ spiralled around the point $\mu_{p/q}$, then our first step is to show that in any open set $U \subseteq \mathbb{R}$, the integers $S_{p/q}$ for $p/q \in U$ are unbounded. We call this spiralling to an indefinite extent. The next step is to show that for certain sequences $p_n/q_n \rightarrow \omega$ with $S_{p_n/q_n} \rightarrow \infty$, the boundary spirals infinitely at the limit point $\lim_{n \rightarrow \infty} \mu_{p_n/q_n}$.

We start out by reviewing some of the relevant theory of hyperbolic geometry in chapter 2. In chapter 3 we define the idea of a slice of $\mathcal{QF}$, and define the Maskit, Bers and Earle slices in

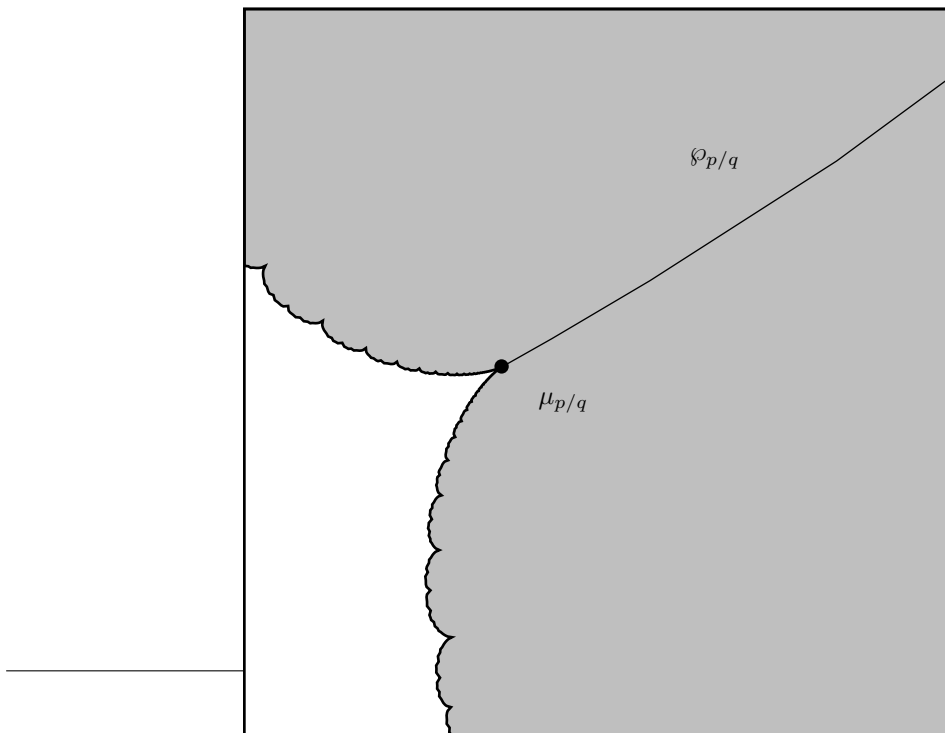


Figure 1.3: Maskit slice with single pleating ray

particular. The original work starts in chapter 4 in which we prove that there is spiralling to an indefinite extent, and infinite spiralling at an uncountable, dense set of points. Chapter 5 contains the conjectural material and numerical evidence, including the conjecture that the boundary spirals infinitely almost everywhere, and conjectures about the Hausdorff dimension of the Maskit slice. Finally, in chapter 6 we discuss the algorithms used in finding this evidence. We hope that this might be of use to anyone else interested in studying this subject experimentally.

Appendix A lists the notation used in this thesis.

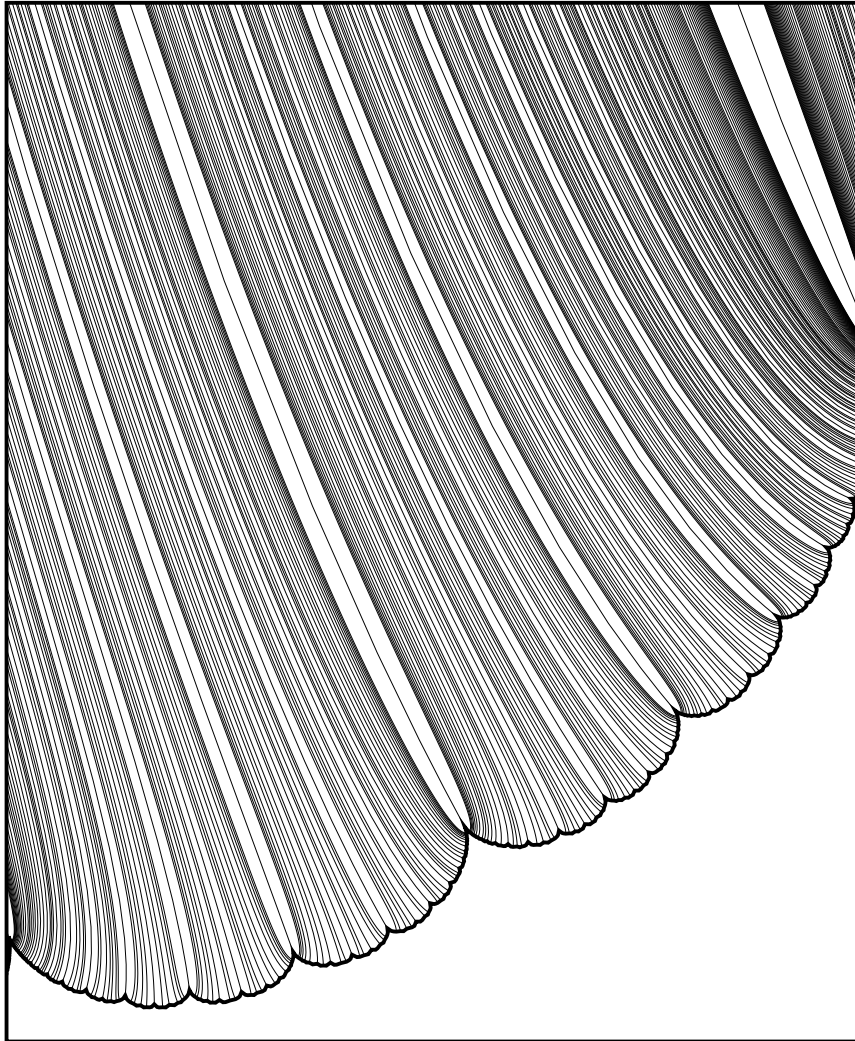


Figure 1.4: Maskit slice with many pleating rays

Chapter 2

Hyperbolic geometry and deformation spaces

2.1 Kleinian groups and quasifuchsian manifolds

A Kleinian group is a discrete subgroup $G \leq \mathrm{PSL}_2\mathbb{C}$. The *regular set*, or *domain of discontinuity* of G is the largest set $\Omega \subseteq \hat{\mathbb{C}}$ on which G acts properly discontinuously. The complement $\Lambda = \hat{\mathbb{C}} - \Omega$ is called the *limit set* of G (see figure 2.1). If G has an invariant disc $\Delta \subseteq \hat{\mathbb{C}}$ then G is a *Fuchsian group* and Λ is contained in the boundary of the disc.

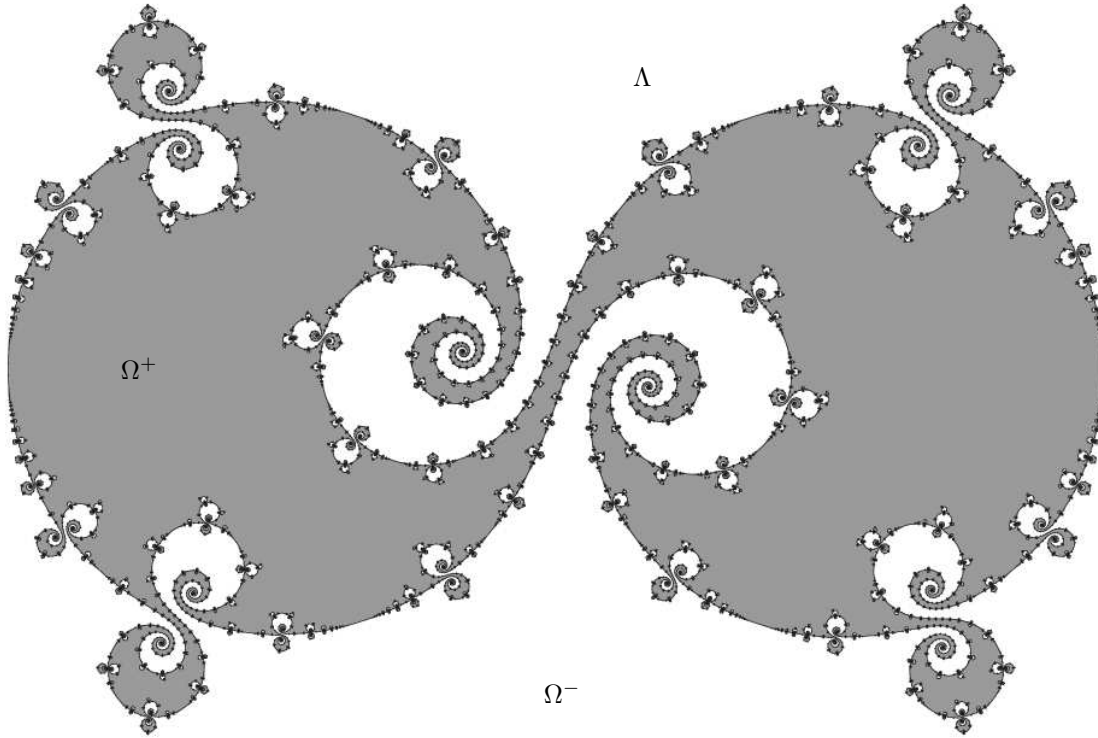


Figure 2.1: Limit set and domains of discontinuity of a once punctured torus group. (Figure courtesy of David Wright.)

For finitely generated G , the quotient manifold Ω/G is a finite union of Riemann surfaces of finite type (see [Ahlfors64] and [Kapovich01]). If Ω/G consists of two once-punctured tori then G is called a *quasifuchsian once-punctured torus group*. In this case, Ω consists of two connected, simply

connected, G -invariant components Ω^+ and Ω^- such that each of $\Omega^\pm/G$ is a once punctured torus. The limit set will be a topological circle separating $\Omega^\pm$. Figure 2.1 illustrates this. The group G will be a free group on two generators $\langle A, B \rangle$, where the commutator $[A, B]$ is parabolic, which we can identify with the fundamental group of a once punctured torus. Write Σ for a fixed once-punctured torus. The group G acts on the hyperbolic upper half space $\mathbb{H}^3$, and we define the manifold

$$M = (\mathbb{H}^3 \cup \Omega)/G.$$

This is a *quasifuchsian manifold* and is homeomorphic to $\Sigma \times [0, 1]$. The boundary has two components $\Omega^\pm/G$ corresponding to $\Sigma \times \{0\}$ and $\Sigma \times \{1\}$. The leftmost manifold, M_1 , in figure 2.2 gives a schematic view of this. See [MatTan98], [Marden74] and [Marden06] for details.

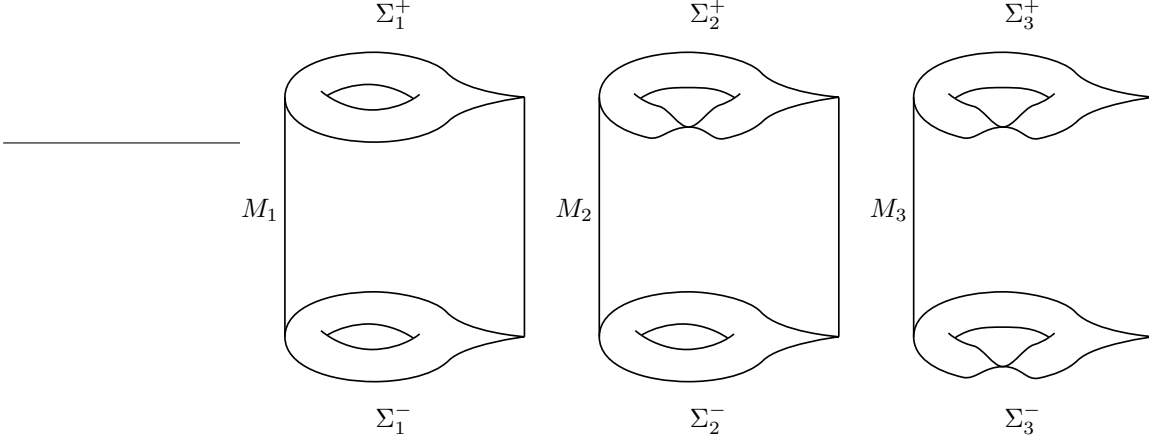


Figure 2.2: Various manifolds

We can consider the group G along with a choice of generators A, B as a discrete representation

$$\rho : \pi_1(\Sigma) = \langle X, Y \rangle \longrightarrow \mathrm{PSL}_2\mathbb{C},$$

such that $\rho([X, Y])$ is parabolic. Here we can think of X and Y as abstract symbols, or as generators of $\pi_1(\Sigma)$.

2.2 Deformation spaces

Thinking of once-punctured torus groups as representations subject to certain constraints gives us a nice way of defining the space of once-punctured torus groups. This *deformation space* is defined as follows. See [Kapovich01], [MatTan98] and [Marden06] for details. First of all, let

$$\Gamma = \pi_1(\Sigma) = \langle X, Y \rangle.$$

Now define

$$\mathcal{R}(\Gamma) = \mathrm{Hom}(\Gamma, \mathrm{PSL}_2\mathbb{C})/\mathrm{PSL}_2\mathbb{C}$$

to be the representation space of Γ modulo conjugation by $\mathrm{PSL}_2\mathbb{C}$. Similarly, we define

$$\mathcal{R}_p(\Gamma) = \{[\rho] \in \mathcal{R}(\Gamma) : \rho([X, Y]) \text{ is parabolic}\}.$$

We define *quasifuchsian space* $\mathcal{QF}$ to be the set of those classes of representations $[\rho] \in \mathcal{R}_p(\Gamma)$ whose images are quasifuchsian once-punctured torus groups.

The Teichmüller space $\mathrm{Teich}(\Sigma)$ of Σ is the space of marked complex structures on Σ with the Teichmüller metric. A marking is just an ordered choice of generators of $\pi_1(\Sigma)$. The Teichmüller metric is defined in terms of quasiconformal maps, but we need only note here that $\mathrm{Teich}(\Sigma)$ is isometric to the hyperbolic upper half plane $\mathbb{H} = \mathbb{H}^2$. See [ImaTan92], [Lehto87] and [Bers70] for details.

Bers' Simultaneous Uniformisation theorem [Bers60] implies that $\mathcal{QF}$ is conformally equivalent to $\text{Teich}(\Sigma) \times \text{Teich}(\bar{\Sigma})$ (where $\bar{\Sigma}$ is Σ with the reverse orientation). Elements of $\mathcal{QF}$ are characterised by the Teichmüller parameters $\nu^\pm$ of the two boundary components $\Omega^\pm/G$.

We can complete $\mathcal{QF}$ to $\overline{\mathcal{QF}}$. We define $\overline{\mathcal{QF}}$ to be the algebraic closure of $\mathcal{QF}$ in $\mathcal{R}_p(\Gamma)$. In a neighbourhood of $\overline{\mathcal{QF}}$, $\mathcal{R}_p(\Gamma)$ is a smooth complex variety of dimension 2 (see [Kapovich01]). In figure 2.2, the manifolds M_2 and M_3 are elements in the boundary of $\overline{\mathcal{QF}}$. See section 2.5 for more details.

Although $\mathcal{QF}$ is conformally equivalent to $\mathbb{H} \times \mathbb{H}$, the completion $\overline{\mathcal{QF}}$ has an extremely complicated structure. Figure 2.3 is a picture of an “exotic slice” of $\overline{\mathcal{QF}}$. You can see that the boundary is not a simple curve. There are now many papers detailing the complicated way in which $\overline{\mathcal{QF}}$ self-intersects, see for example [BromHolt01].

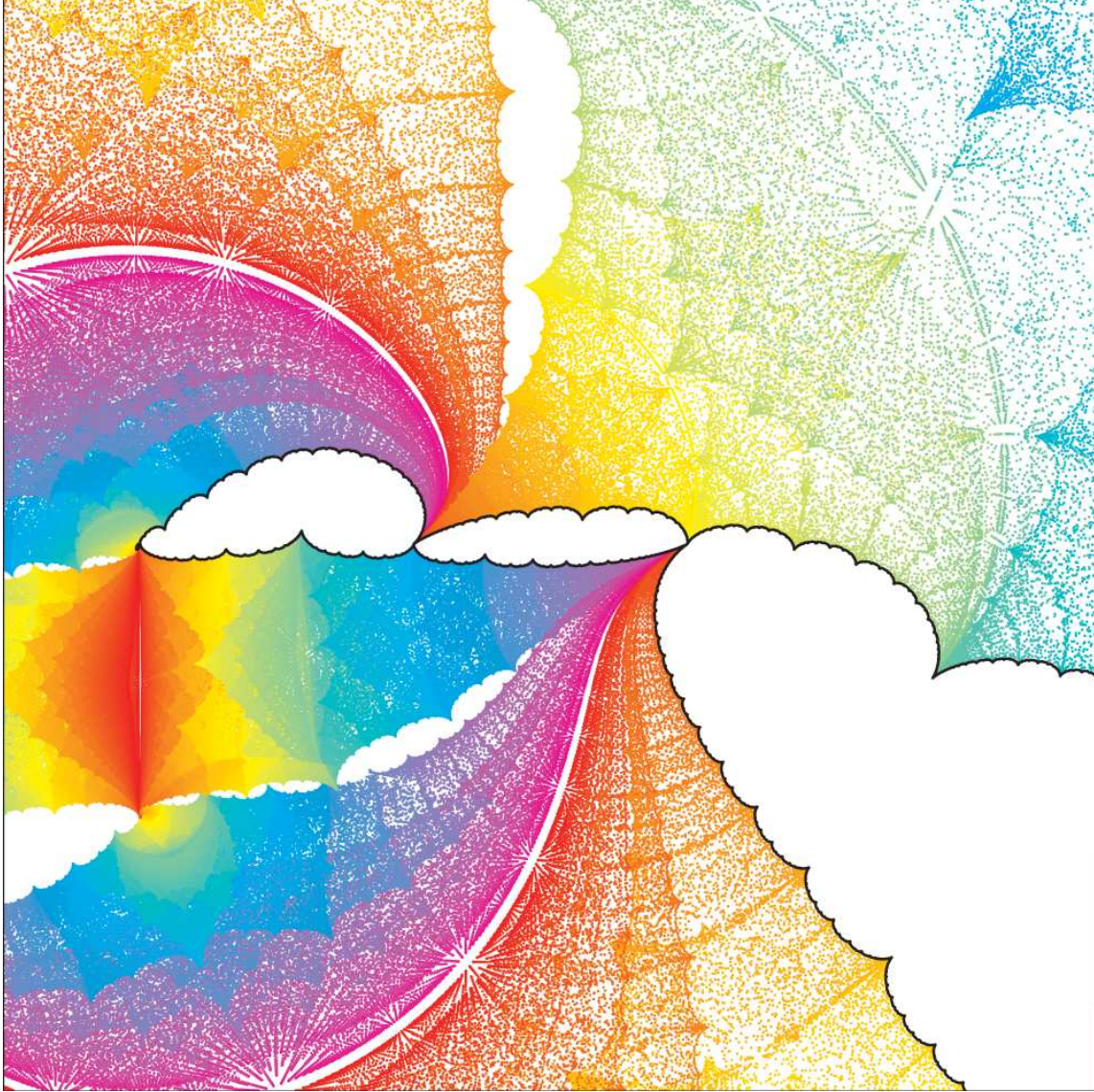


Figure 2.3: An exotic slice of $\mathcal{QF}$. The large white regions are outside $\mathcal{QF}$, the regions filled with dots are inside. (Figure courtesy of David Wright.)

2.3 Combinatorics

2.3.1 Farey series

The Farey series and Farey graph are the foundation of the combinatorics of the Maskit and Bers slices, which we will define later. We define $\hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ (and similarly $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$). Rational numbers p/q and r/s satisfying $ps - rq = \pm 1$ are said to be *Farey neighbours*. For such fractions we define the operation of *Farey addition* $\oplus$ by

$$\frac{p}{q} \oplus \frac{r}{s} = \frac{p+r}{q+s}.$$

The Farey graph, embedded in the hyperbolic upper half plane model in figure 2.4, has vertex set $\hat{\mathbb{Q}}$, and two vertices are connected by an edge if they are neighbours. If p/q and r/s are neighbours, then p/q , r/s and $p/q \oplus r/s$ are the vertices of a triangle in the Farey graph. As mentioned in the introduction, cusp points correspond to rational numbers p/q . Two cusps whose corresponding rationals are Farey neighbours will be said to be *neighbouring cusps*.

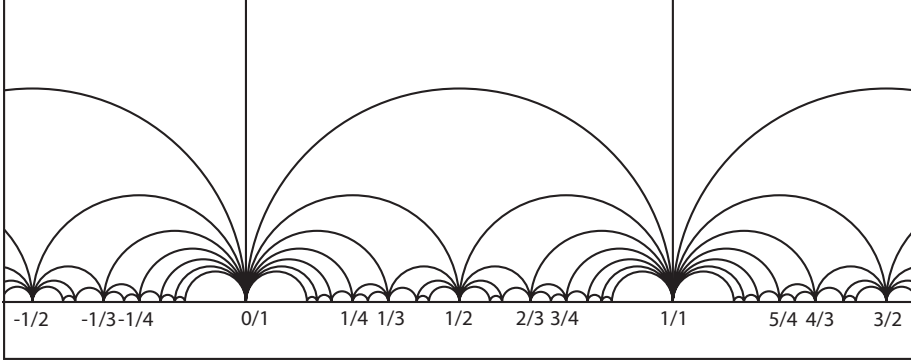


Figure 2.4: The Farey graph

2.3.2 The curve complex

We have already defined Σ to be a fixed once-punctured torus, now define S to be a fixed torus, say $\Sigma = S - \{*\}$. Now both $\pi_1(S)$ and $\pi_1(\Sigma)$ are generated by X and Y . The group $\pi_1(S)$ is the free Abelian group $\mathbb{Z}^2$, whereas, as we have noted, $\pi_1(\Sigma) = \langle X, Y \rangle$. Every free homotopy class of a simple closed curve on S or Σ is represented by an element of the fundamental group corresponding to an element of $\hat{\mathbb{Q}}$. On S , it is represented by the element $X^q Y^p$. On Σ , the same curve is represented by the word $W_{p/q}$ defined inductively below. This will be very important throughout the rest of this thesis.

$$\begin{aligned} W_{0/1} &= X \\ W_{1/0} &= Y \\ W_{-1/0} &= Y^{-1} \\ W_{p/q \oplus r/s} &= W_{p/q} W_{r/s} && \text{if } ps - rq = -1 \\ W_{p/q \oplus r/s} &= W_{r/s} W_{p/q} && \text{if } ps - rq = 1 \end{aligned}$$

See [Wright88], [KeenSeries93] and [Series85] for more details. Essentially, to find the word $W_{p/q}$ you do the following (see figure 2.5). Draw an integral grid in the plane. Now draw a straight line of gradient p/q through the origin. Each time it intersects a vertical line of the grid you write X and each time it intersects a horizontal line of the grid you write Y . You continue until you get to an integral point. This corresponds to finding a simple, closed curve on Σ in the same homotopy class as $X^q Y^p$ on S . The inductive definition above gives a conjugate of this word, but the algebraic properties of the specific word $W_{p/q}$ are important (because of the trace identities described in

section 6.2.1).

Figure 2.5 demonstrates this for $W_{2/3}$. Reading off the word generated by that method gives XYX^2Y . This turns out to be a conjugate (a cyclic reordering) of $W_{p/q}$. It is easy to see that you can cyclically reorder because you can move your starting point without changing the free homotopy class. For example, from the origin to the point $(5/3, 10/9)$, between the first Y and the second X .

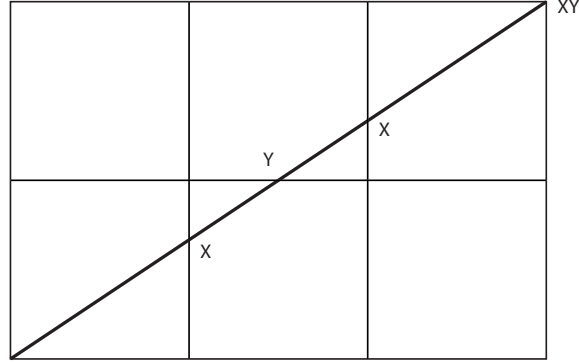


Figure 2.5: Finding $W_{2/3}$ on the once-punctured torus using gradients

We can compute $W_{2/3}$ using the recursive formula as follows. Firstly, $W_{1/1} = W_{0/1}W_{1/0}$ as $0/1 \oplus 1/0 = 1/1$, so $W_{1/1} = XY$. Now $W_{1/2} = W_{0/1}W_{1/1} = X^2Y$. Finally $W_{2/3} = W_{1/2}W_{1/1} = X^2YXY$. This process is illustrated in figure 2.6. Each edge of the graph intersected by the vertical line ending at $2/3$ is a step in the inductive definition of $W_{2/3}$ (see [Series85] for more on this).

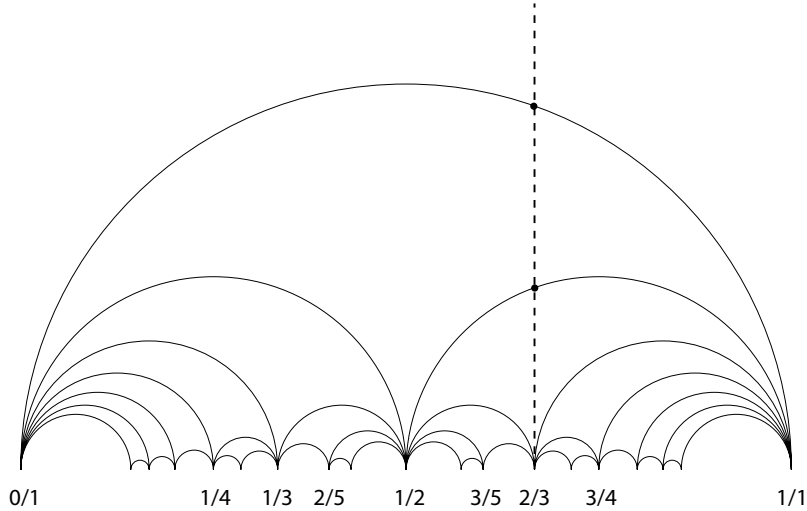


Figure 2.6: The Farey graph, showing how to get to $2/3$

2.4 Pleating invariants

2.4.1 The convex core

The convex core of M is the smallest hyperbolic closed set containing all the closed geodesics in M . An equivalent way to define it is to define $\mathcal{C}$ to be the hyperbolic convex hull of the limit set Λ in $\mathbb{H}^3$. The projection $\mathcal{C}/G$ is the convex core of M . If G is Fuchsian then $\mathcal{C}$ will be a hyperbolic plane, otherwise if G is a quasifuchsian once-punctured torus group then $\mathcal{C}$ will be a three-dimensional subset with two boundary components $\partial\mathcal{C}^\pm$ facing on to $\Omega^\pm$. We consider only the case of G quasifuchsian. See [EMM03], [EMM03a], [EpsMar06], and [Thurston80] for more details.

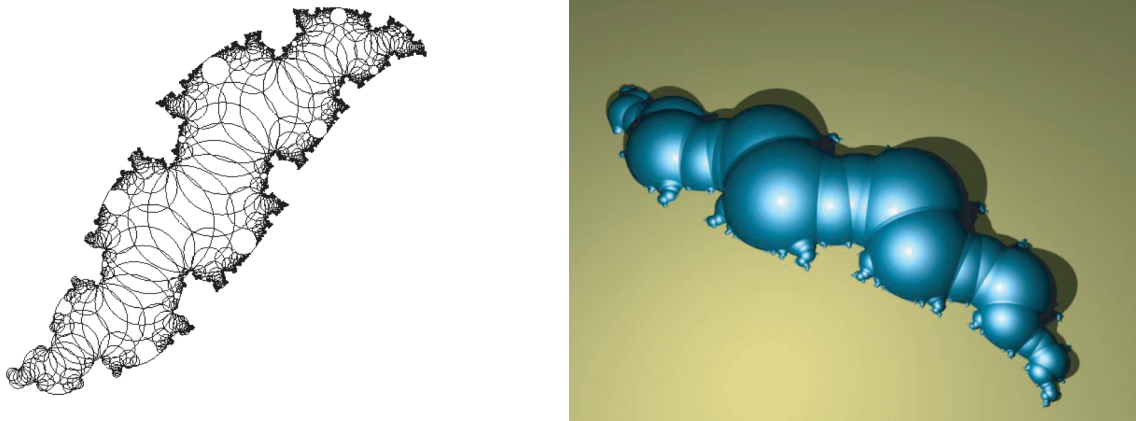


Figure 2.7: Convex hull boundary (right) of a once punctured torus group with limit set (left). (Figure courtesy of Yair Minsky.)

The boundary of $\mathcal{C}$ consists of totally geodesic faces separated by geodesic *bending lines* (see the right hand side of figure 2.7). The set of bending lines is the *bending locus*. This locus projects to a *geodesic lamination* on $\partial\mathcal{C}^\pm/G$, with a *transverse measure* $\text{pl}^\pm(G)$. A geodesic lamination is a closed, disjoint set of geodesics (called the *leaves* of the lamination). A transverse measure is a measure on curves transverse to the lamination. In the case where the geodesic lamination consists only of closed geodesics, we think of a transverse measure as just assigning a weight to each geodesic. The surface $\partial\mathcal{C}^\pm/G$ is homeomorphic to Σ , and carries a natural pleated surface structure from the hyperbolic metric on M . Here, we need only note that for points not on a bending line, there is a local hyperbolic metric, and that there is a hyperbolic metric along the bending lines. We will use $\text{pl}^\pm(\rho)$ to mean $\text{pl}^\pm(\rho(\Gamma))$ and so forth. The right hand side of figure 2.7 shows the convex hull boundary of the limit set on the left hand side. See [KeenSeries04] for more details and [Series99] for a very clear exposition.

2.4.2 Rational pleating varieties

If the bending locus projects to a single closed curve on $\partial\mathcal{C}/G$, then this curve will be represented by p/q for some $p/q \in \hat{\mathbb{Q}}$. In fact, this curve will be the projection of the axis of $W_{p/q}$. In this case, we say that it is a *rational pleating locus*. The transverse measure $\text{pl}^\pm(G)$ simply assigns a weight to the curve corresponding to the angle between the two planes in $\partial\mathcal{C}$ on either side of it.

Let $\gamma_{p/q}$ be the geodesic on $\partial\mathcal{C}^\pm/G$ corresponding to the word $W_{p/q}$. We write $|\text{pl}^\pm|$ to mean the support of the transverse measure $\text{pl}^\pm$. The $(p/q, r/s)$ -*pleating variety* is defined to be the set of elements of $\mathcal{QF}$ which have the p/q and r/s geodesics as their bending loci. That is, the pleating variety is defined by

$$\mathcal{P}_{p/q, r/s} = \{[\rho] \in \mathcal{QF} : |\text{pl}^+(\rho)| = \gamma_{p/q}, |\text{pl}^-(\rho)| = \gamma_{r/s}\}.$$

2.4.3 Pleating invariants and coordinates

A measured geodesic lamination is just a geodesic lamination together with a transverse measure on it. The space of measured geodesic laminations on a hyperbolic surface S is written $\mathcal{ML}(S)$. Note that the set $\mathcal{ML}(S)$ does not depend on the hyperbolic metric on S (see references in [KeenSeries04]). The space of projective classes is written $\mathcal{PML}(S)$, and if S is a once-punctured torus it is homeomorphic to S^1 . We define the lamination length ℓ_μ of a measured geodesic lamination $\mu \in \mathcal{ML}$ to be the total mass of S for the measure given by the product of the hyperbolic length along the leaves of the geodesic lamination $|\mu|$ with the transverse measure μ . See [CEG87], [Bonahon01] and [Thurston80] for more details. Given a group G we can identify $\partial\mathcal{C}^\pm/G$ with Σ and therefore $\mathcal{ML}(\partial\mathcal{C}^\pm/G)$ with $\mathcal{ML}(\Sigma)$. The quantity ℓ_μ for $\mu \in \mathcal{ML}(\Sigma)$ depends on G via the pleated surface structure on $\partial\mathcal{C}^\pm/G$. If $\mu' = c\mu$ for some $c > 0$ then $\ell_{\mu'} = c\ell_\mu$. If G is not Fuchsian (that

is, the components $\Omega^\pm$ are not round disks), then the projective classes of $(\mu^\pm, \ell_{\mu^\pm})$, for any choice of $\mu^\pm \in [\text{pl}^\pm(G)]$, are the *pleating invariants* of G . Keen and Series prove in [KeenSeries04] that a non-Fuchsian marked punctured torus group is determined uniquely up to conjugacy in $\text{PSL}_2\mathbb{C}$ by its pleating invariants.

If the axis of $W \in G$ is a bending line of $\partial\mathcal{C}^\pm/G$ then $\text{Tr}(W) \in (-\infty, -2) \cup (2, \infty)$. See [KeenSeries93] for the proof, but in brief this follows because W fixes $\partial\mathcal{C}^\pm$ and the two planes whose intersection is the axis of W , meaning that W must be purely hyperbolic. The *complex length* $\lambda(W)$ of W is defined to satisfy

$$\text{Tr}(W) = 2 \cosh \lambda(W)/2,$$

with the additional requirement that $\text{Re } \lambda(W) > 0$ and $\text{Im } \lambda(W) \in (-\pi/2, \pi/2)$. We write $\ell(W)$ for $\text{Re } \lambda(W)$, this is the *translation length* of W . If the axis of W is a bending line then $\lambda(W) \in \mathbb{R}^{>0}$.

Define $\delta_\gamma \in \mathcal{ML}$ to be the transverse measure on Σ which simply assigns the weight 1 to the geodesic γ . If $|\text{pl}^\pm(G)| = \gamma$, with γ the geodesic corresponding to W , then we can write $\text{pl}^\pm(G) = \theta\delta_\gamma$, where θ is the angle between the pleating planes that join at γ . Now $\ell_{\delta_\gamma} = \ell(W)$, which is also the hyperbolic length of the geodesic γ . The projective pair $(\delta_\gamma, \ell_{\delta_\gamma})$ is the same as that of the pair $(\text{pl}^\pm(G), \ell_{\text{pl}^\pm(G)})$. This defines coordinates

$$\phi : \wp_{p/q, r/s} \longrightarrow \mathbb{R}^{>0} \times \mathbb{R}^{>0}; [\rho] \mapsto (\ell(\rho(W_{p/q})), \ell(\rho(W_{r/s})))$$

for the rational pleating variety $\wp_{p/q, r/s}$ (see [KeenSeries04]). Moreover, the set of rational pleating varieties is dense in $\mathcal{QF}$ (see [KeenSeries04]), and the complex lengths $(\lambda(W_{p/q}), \lambda(W_{r/s}))$ are smooth local coordinates at any point in a neighbourhood of $\wp_{p/q, r/s}$ in $\mathcal{QF}$ (see [ChoiSeries06]).

Suppose you chose a path $\rho : (0, \infty) \rightarrow \wp_{p/q, r/s}$ such that $\ell(W_{p/q}(\rho(t))) = t$ and $\ell(W_{r/s}(\rho(t)))$ is fixed. The limit of $\rho(t)$ as $t \rightarrow 0$ is an element $\rho(0) \in \partial\mathcal{QF}$. The $+$ end of the corresponding manifold will be a triply punctured sphere (like manifold M_2 in figure 2.2). As t gets closer to 0, the length of $\gamma_{p/q}$ gets closer to 0 until finally in the limit it disappears. We say that $\gamma_{p/q}$ has been *pinched* to a point. We say that the corresponding manifold M has a cusp corresponding to $\gamma_{p/q}$.

2.5 End invariants

We will also use another set of coordinates for $\mathcal{QF}$. If $[\rho] \in \overline{\mathcal{QF}}$ then any component of Ω can be reached either by going to the $+$ end or the $-$ end of M , this divides Ω into two G -invariant subsets $\Omega^\pm$. There are three possibilities for each of $\Omega^\pm$, and the definition of the corresponding *end invariant* $\nu^\pm$ is as follows.

1. $\Omega^\pm$ is a topological disc, $\Omega^\pm/G$ is a once-punctured torus. In this case we define $\nu^\pm$ to be the Teichmüller parameter of $\Omega^\pm/G$. Manifold M_1 in figure 2.2 has both ends of this type.
2. $\Omega^\pm$ is an infinite union of discs, $\Omega^\pm/G$ is a triply-punctured sphere obtained from the boundary of $\Sigma \times (0, 1)$ by deleting a curve $\gamma_\pm$. This curve corresponds to an element $W_{p/q}$ and in this case we define $\nu^\pm = p/q$. This is the case where the curve $\gamma_{p/q}$ has been pinched discussed above. Manifold M_2 in figure 2.2 has one end of this type, and one end of the first type. Manifold M_3 has both ends of this type.
3. $\Omega^\pm$ is empty. This case can be considered a limit of cases where $\nu^\pm = p/q$ and we get $\nu^\pm \in \mathbb{R} - \mathbb{Q}$.

Writing $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\} = S^1$, we define $\overline{\mathbb{H}} = \mathbb{H} \cup \hat{\mathbb{R}}$ (or equivalently $\overline{\mathbb{H}}$ is a closed disc), and Δ to be the diagonal of $\hat{\mathbb{R}} \times \hat{\mathbb{R}}$. For any representation $[\rho] \in \overline{\mathcal{QF}}$ we can assign its pair of end invariants $(\nu_-, \nu_+) \in (\overline{\mathbb{H}} \times \overline{\mathbb{H}}) - \Delta$. There is a continuous bijection

$$\nu^{-1} : (\overline{\mathbb{H}} \times \overline{\mathbb{H}}) - \Delta \longrightarrow \overline{\mathcal{QF}}$$

(but the inverse map ν is not even continuous). If a marked punctured torus group is in $\partial\mathcal{QF} = \overline{\mathcal{QF}} - \mathcal{QF}$ then either $\nu^\pm \in \mathbb{Q} \cup \{\infty\}$ corresponding to pinching a curve on $\Omega^\pm/G$ of slope $\nu^\pm$ to a point, or $\nu^\pm$ is an irrational real. See [Minsky99], [Bonahon86] and [Thurston80] for more details.

Chapter 3

Slices of quasifuchsian space

A slice of quasifuchsian space is a one-complex dimensional subset of $\overline{\mathcal{QF}}$. The prototypical slice is a *Bers slice*. This is the set of elements of $\overline{\mathcal{QF}}$ with a fixed once-punctured torus structure on one or other of the two ends of M . In terms of end invariants, a Bers slice $\mathcal{B}$ is just $\nu^{-1}(\{*\} \times \overline{\mathbb{H}} - \{*\})$ or $\nu^{-1}(\overline{\mathbb{H}} - \{*\} \times \{*\})$ where $*$ $\in \mathbb{H}$. The interior of a Bers slice consists of manifolds of type M_1 in figure 2.2, and the boundary consists of manifolds of type M_2 .

A *Maskit slice*, which we will occasionally denote $\mathcal{M}$, is a slice of the boundary of $\mathcal{QF}$, and is defined in the same way as the Bers slice except that we require $*$ $\in \hat{\mathbb{Q}}$. These are sometimes referred to as *rational Maskit slices*, and sometimes as *limit Bers slices*. The interior of a Maskit slice consists of manifolds of type M_2 in figure 2.2, and the boundary consists of manifolds of type M_3 .

We will primarily be concerned with Maskit and Bers slices, with particular emphasis on the former as they have a nice algebraic structure.

Other slices are the Earle slice in which the two ends of M are required to have end invariants satisfying $\nu^+ \nu^- = 1$; the lambda slices; etc.

For slices of $\mathcal{QF}$, a pleating ray is the one-complex dimensional analogue of the pleating varieties discussed in section 2.4.2. In the notation of that section we define

$$\wp_{p/q} = \{[\rho] \in \mathcal{QF} : |\text{pl}^+(\rho)| = \gamma_{p/q} \text{ or } |\text{pl}^-(\rho)| = \gamma_{p/q}\}.$$

The p/q -pleating ray for a slice is the intersection of $\wp_{p/q}$ with the slice. Abusing notation, this pleating ray will also be written $\wp_{p/q}$. These rays can be seen in figures 3.2, 4.5 and 4.8.

Consider the function

$$\text{Tr}_W : \mathcal{QF} \longrightarrow \mathbb{C}/\{\pm 1\}; [\rho] \mapsto \text{Tr}(\rho(W)).$$

If $[\rho]$ is in some slice of $\mathcal{QF}$ and $[\rho] \in \wp_{p/q}$ then it satisfies

$$\text{Tr}_{W_{p/q}}([\rho]) \in (-\infty, -2) \cup (2, \infty).$$

However, not all $[\rho]$ satisfying this trace condition will be in the pleating ray. The locus of points satisfying $\text{Tr}_{W_{8/21}} \in \mathbb{R}$ for the Maskit slice is shown in figure 3.1.

3.1 Cusps

A *cusp* in the boundary of $\mathcal{QF}$ is a point where some element of G has become parabolic (sometimes called *accidentally parabolic*). That is, an element $[\rho] \in \mathcal{QF}$ such that $\rho(W)$ is parabolic for some $W \in \Gamma$. If there is a generating pair $W_1, W_2 \in \Gamma$ such that $\rho(W_1)$ and $\rho(W_2)$ are both parabolic then $[\rho]$ is called a *double cusp*.

Cusps are dense in the boundary of $\mathcal{QF}$ (see [McMullen91]) and getting computers to systematically find cusps turns out to be a very good way of drawing pictures of slices of $\mathcal{QF}$ (although not the only way). They are connected to pleating rays in that at the end point of a pleating ray, the corresponding word $W_{p/q}$ is parabolic, as $\text{Tr}_{W_{p/q}}([\rho]) = \pm 2$. This fact leads to an algorithm,

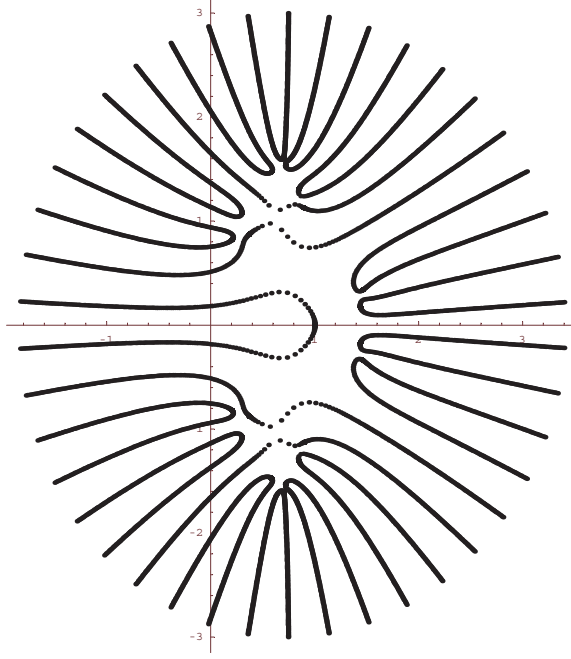


Figure 3.1: Real locus of $\text{Tr}_{W_{8/21}}$ in the Maskit slice. There are 21 smooth, connected subsets, each tending to ∞ at both ends.

discussed later, for finding cusps.

3.2 Embeddings

In order to be able to draw pictures of a slice, you need to embed it in $\mathbb{C}$. There are standard ways of doing this for various slices.

3.2.1 Maskit

The Maskit slice (see [Maskit74]), can be very simply embedded in $\mathbb{C}$ in the following way. Define

$$g : \mathbb{C} \longrightarrow \text{Hom}(\Gamma, \text{SL}_2\mathbb{C})$$

by

$$g(\mu)(X) = -i \begin{pmatrix} \mu & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad g(\mu)(Y) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

Let $\pm : \text{SL}_2\mathbb{C} \rightarrow \text{PSL}_2\mathbb{C}$ be the quotient map. The map $[g] : \mathbb{C} \rightarrow \mathcal{R}_p(\Gamma)$ is defined by setting $[g](\mu) = [\pm g(\mu)]$, and the set $\overline{\mathcal{M}} = [g]^{-1}(\overline{\mathcal{QF}}) \cap \mathbb{H}$ is shown in figure 3.2. The jagged curve at the bottom is the boundary, and continues periodically in the same fashion with period 2. The slice itself consists of the points on the upper side of this curve. The almost vertical curves coming out of points on the boundary curve are the pleating rays. See [KeenSeries93] and more generally [KeenSeries04] for more details.

You can see in figure 3.2 that none of the pleating rays intersect any of the other pleating rays. This is true, and furthermore the pleating rays are dense in $\mathcal{M}$. In fact, you can even use pleating rays to define *pleating coordinates*. If $\mu \in \wp_{p/q}$ and $\text{Re } \lambda(W_{p/q}(\mu)) = \ell$ say then write $M(\mu) = (p/q, \ell/q)$. This extends by continuity to a homeomorphism $M : \mathcal{M} \rightarrow \mathbb{R} \times \mathbb{R}^{>0}$. The (rational) pleating rays are interpolated by irrational pleating rays $M^{-1}(\{\omega\} \times \mathbb{R}^{>0})$ for irrational ω , which correspond to irrational bending laminations. Horizontal and vertical lines for these coordinates are shown in

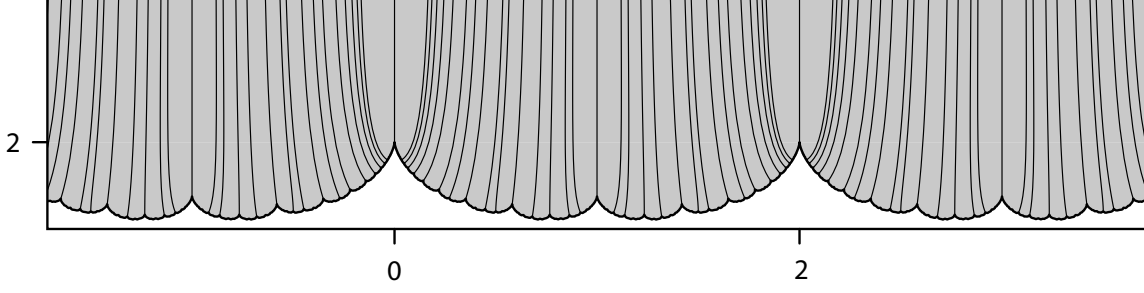


Figure 3.2: The Maskit slice, with pleating rays and interior shaded grey

figures 3.3 and 3.4. See section 2.4, and [KeenSeries93] for more details.

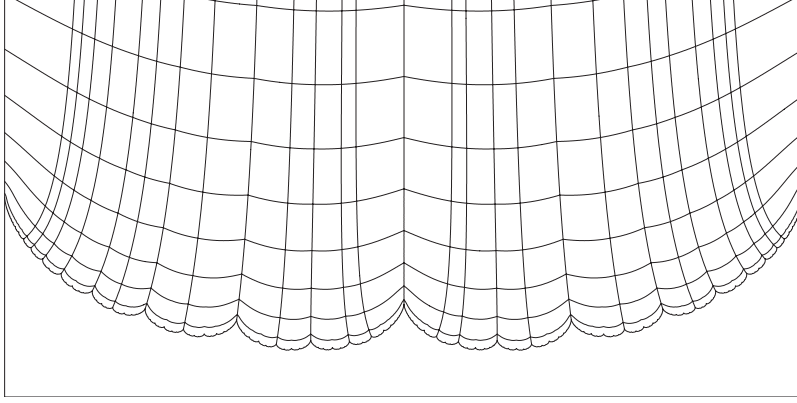


Figure 3.3: The Maskit slice, with pleating coordinates shown. (Figure courtesy of David Wright.)

3.2.2 Bers

We do not give the full details of the Bers embedding because they are not relevant to the results in this thesis. See [Bers70], [Maskit70], [KomSug04] and [Tanigawa97] for more details.

A complex projective structure on a surface S is an atlas of charts whose transition maps are restrictions of elements of $\mathrm{PSL}_2\mathbb{C}$ acting as Möbius transformations on $\mathbb{C}$. We write $P(S)$ for the space of complex projective structures on S . We consider $P(\Sigma)$. Given a complex projective structure on Σ we get a holonomy map $\rho : \Gamma \rightarrow \mathrm{PSL}_2\mathbb{C}$. Taking holonomy gives us a map $P(\Sigma) \rightarrow \mathcal{R}_p(\Gamma)$.

On the other hand taking the Schwarzian derivative of the developing map of a projective structure gives us a quadratic differential ϕ . This identifies the space $P(\Sigma)$ with the space B_2 of quadratic differentials. This space has complex dimension 1. Identifying $\mathbb{C}$ with B_2 , B_2 with $P(\Sigma)$ and using the holonomy map we get a map $\chi : \mathbb{C} \rightarrow \mathcal{R}_p(\Gamma)$. The connected component of $\chi^{-1}(\mathcal{QF})$ containing 0 is the Bers slice $\mathcal{B}$.

Figure 3.5 shows the Bers slice in $\mathbb{C}$.

3.2.3 Earle

The Earle slice is defined by imposing certain symmetries on elements of $\mathcal{QF}$. We give an explicit embedding of the Earle slice in $\mathbb{C}$, and then discuss the symmetries.

As in the case of the Maskit slice, we give an explicit map $g : \mathbb{C} - \{0\} \rightarrow \mathrm{Hom}(\Gamma, \mathrm{SL}_2\mathbb{C})$ by

$$g(d)(X) = \begin{pmatrix} \frac{d^2+1}{d} & \frac{d^3}{2d^2+1} \\ \frac{2d^2+1}{d} & d \end{pmatrix}, \quad \text{and} \quad g(d)(Y) = \begin{pmatrix} \frac{d^2+1}{d} & -\frac{d^3}{2d^2+1} \\ -\frac{2d^2+1}{d} & d \end{pmatrix}.$$

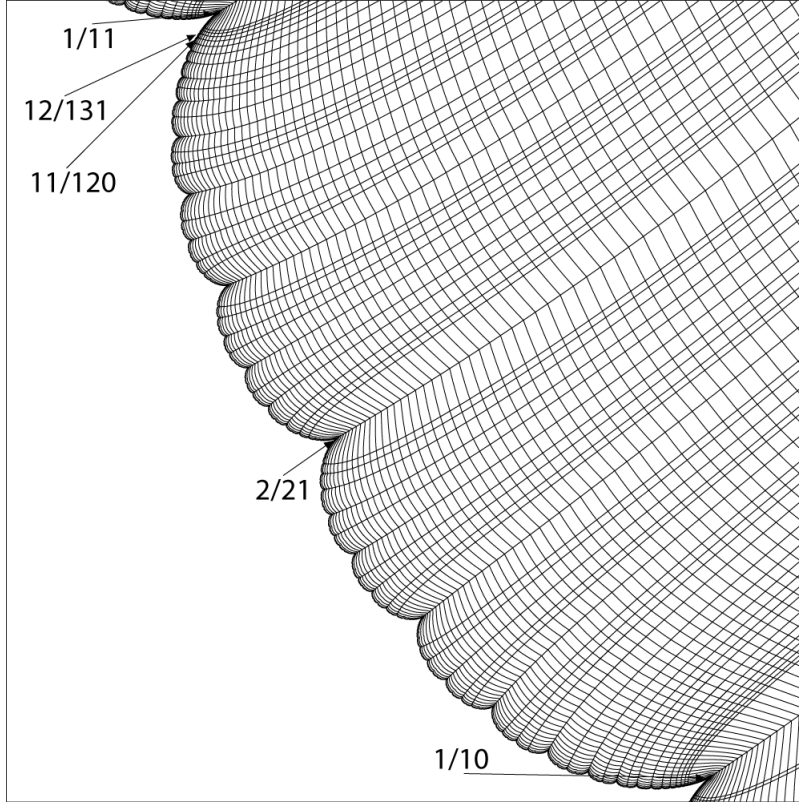


Figure 3.4: Detail from the Maskit slice, with pleating coordinates shown. (Figure courtesy of David Wright.)

As before, let $\pm : \mathrm{SL}_2\mathbb{C} \rightarrow \mathrm{PSL}_2\mathbb{C}$ be the quotient map and define the map $[g] : \mathbb{C} \rightarrow \mathcal{R}_p(\Gamma)$ by setting $[g](d) = [\pm g(d)]$. The Earle slice is illustrated in figure 3.6.

Write $\theta : \Gamma \rightarrow \Gamma$ for the involution $\theta(X) = Y$ and $\theta(Y) = X$, and $\Theta : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ for the involution $\Theta(z) = -z$. For the Earle slice as defined above, we have the symmetry that for any $\gamma \in \Gamma$, and any $z \in \hat{\mathbb{C}}$, $\Theta(\gamma z) = \theta(\gamma)\Theta(z)$. In particular, writing $A = g(d)(X)$ and $B = g(d)(Y)$, we have that $A(z) = -B(-z)$. We also have that Θ restricts to a conformal homeomorphism $\Omega^\pm \rightarrow \Omega^\mp$. These symmetries can be made into a general definition for a class of Earle slices $\mathcal{E}_\theta$ defined by any involution θ on Γ . For this particular choice of parameterisation, we also get that the Earle slice is symmetric with respect to the maps $\sigma(d) = \bar{d}$ and $\sigma(d) = 1/2d$, and that $\mathrm{Tr} A = \mathrm{Tr} B$. This last fact is related to another way of looking at the Earle slice. That is, if you embed $\mathcal{QF}$ in $\mathbb{C}^3$ by sending $[\rho]$ to $(\mathrm{Tr} \rho(X), \mathrm{Tr} \rho(Y), \mathrm{Tr} \rho(XY))$, then the Earle slice becomes $\mathcal{QF} \cap \{(x, y, z) : x = y\}$.

See [KomSer01] for more details about the Earle slice.

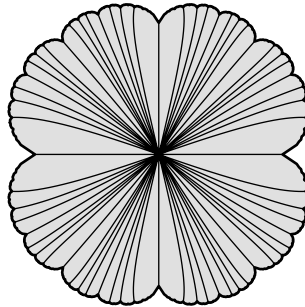


Figure 3.5: The Bers slice, with the inside shaded grey and pleating rays shown. (Figure courtesy of Yohei Komori and Toshiyuki Sugawa.)

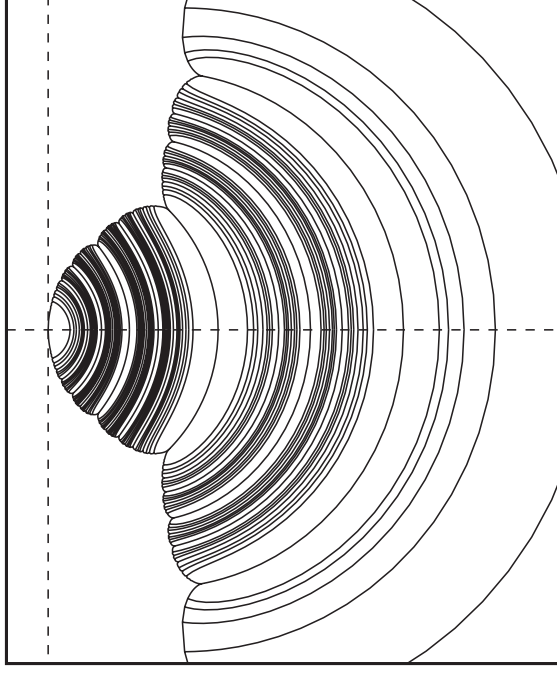


Figure 3.6: The Earle slice, with pleating rays. (Figure adapted from Peter Liepa.)

3.2.4 General setup

To cover the case of the Maskit and Bers slices in one, we define a slice of $\overline{\mathcal{QF}}$ by choosing an injective holomorphic map $f : \mathbb{C} \rightarrow \mathcal{R}_p(\Gamma)$. A certain open subset $K \subseteq \mathbb{C}$ will consist of all the interior points of the slice, so that f maps $\overline{K}$ biholomorphically to either the Maskit or Bers slice contained in $\mathcal{R}_p(\Gamma)$. In both cases, the boundary ∂K is a simple curve, and $\overline{K}$ is simply connected (see [Minsky99]). Abusing language somewhat, we will also sometimes refer to the set K or $\overline{K}$ as the slice without mentioning f .

3.3 Cusp shape

In [Wright88], David Wright showed, on the basis of conjectures now proven (with one exception, about which more later), that for a cusp p/q in the Maskit slice, the set of cusp points corresponding to Farey neighbouring fractions is approximately $(2, 3)$ -cuspidal. More precisely, if p/q and r/s are Farey neighbours, and $r_n = (np + r)/(nq + s) \rightarrow r_\infty = p/q$, define μ_n to be the cusp point corresponding to r_n and μ_∞ to be p/q (see figure 3.7). Then he showed that

$$\mu_n = \mu_\infty - \frac{\pi^2}{A_1 n^2} + \frac{\delta \pi^2 i}{A_1 n^3} (2 + B_0) + O\left(\frac{1}{n^4}\right) \quad (3.1)$$

where A_1 , B_0 and δ are constants. In particular, $\operatorname{Re}(A_1(\mu_n - \mu_\infty)) = -\pi^2/n^2 + O(n^{-3})$, $\operatorname{Im}(A_1(\mu_n - \mu_\infty)) = C/n^3 + O(n^{-4})$ (for a real constant C), so that $z_n = A_1(\mu_n - \mu_\infty)$ approximately satisfies an equation of the form $a \operatorname{Re}(z_n)^3 = \operatorname{Im}(z_n)^2$. A curve of points (x, y) is a $(2, 3)$ -cusp if it satisfies $ax^3 = y^2$.

In [Miyachi03], Hideki Miyachi showed that for the Earle, Maskit and Bers slices, there is a neighbourhood of a cusp in which not only the set of neighbouring cusp points, but the whole boundary curve is approximately $(2, 3)$ -cuspidal. More precisely, he showed that the boundary of any of these slices in the neighbourhood of a cusp is $(2, 3)$ -cuspidal in the following sense.

Definition 3.3.1 *A $(2, 3)$ -cusp curve is a possibly translated and scaled copy of the graph $x^2 = y^3$ in $\mathbb{R}^2$. The cusp point is the translated image of $(0, 0)$. A curve γ is $(2, 3)$ -cuspidal at a point $P \in \gamma$*

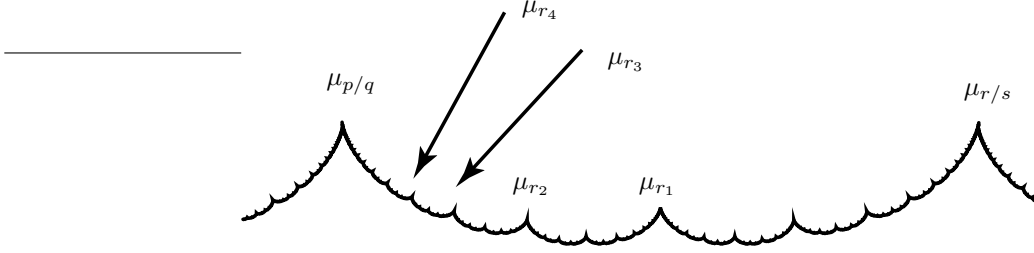


Figure 3.7: Enumerating neighbouring cusps

if in a neighbourhood of P there are two $(2,3)$ -cusp curves with cusp points at P such that γ is contained in the region between the two curves. Figure 3.8 shows this in the case of the boundary of a slice, the two $(2,3)$ -cusps are shown dashed, and the boundary curve γ is unbroken.

Figure 3.9 shows an actual plot of a neighbourhood of a cusp in the Maskit slice (with pleating rays).

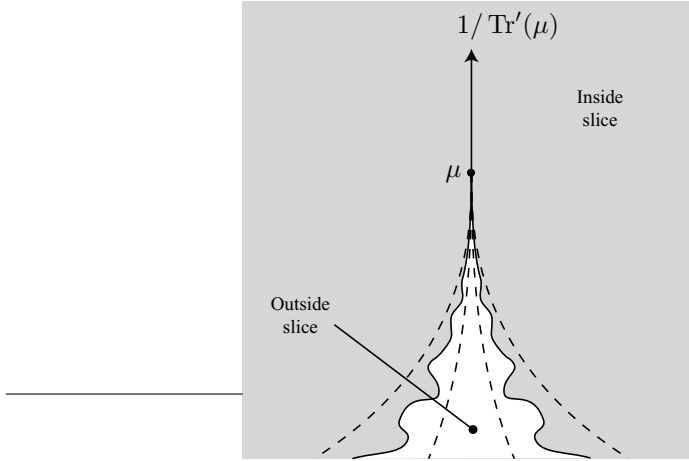


Figure 3.8: Local structure of the boundary near a cusp

3.4 Trace functions

For a word $W \in \Gamma$ and a slice $f : \mathbb{C} \rightarrow \mathcal{R}_p(\Gamma)$, the trace function $\text{Tr}_W : \mathbb{C} \rightarrow \mathbb{C}$ is defined by $\text{Tr}_W(\mu) = \text{Tr } f(\mu)(W)$. In fact, this only defines Tr_W up to ± 1 but if $\mu \in \overline{K}$ then you can choose a sign for Tr_W in a neighbourhood of $f(\mu)$ (or equivalently in a neighbourhood of μ) because $\text{Tr } f(\mu)(W) \neq 0$ for a free representation $f(\mu)$. The quantity Tr_W^2 is always well defined. If W corresponds to a cusp and μ_W is the cusp point on ∂K corresponding to W , then $\text{Tr}_W^2(\mu_W) = 4$. The pleating ray corresponding to W is the unique connected subset $\wp_W \subseteq K$ of $(\text{Tr}_W^2)^{-1}((4, \infty))$ ending at the point μ_W . If we choose a sign for Tr_W in a neighbourhood of μ_W and parameterise $\wp_W$ by $\psi : [2, \infty) \rightarrow \mathbb{C}$ so that $\text{Tr}_W \psi(t) = t$ and differentiate, we get $\psi'(t) \text{Tr}'_W(\psi(t)) = 1$. In particular, the initial direction of $\wp_W$ is $\psi'(2) = 1/\text{Tr}'_W(\mu_W)$. This is also the direction in which the approximately cuspidal boundary points (see section 4.1).

It is proved in [Miyachi03], and more generally in [ChoiSeries06], that:

Proposition 3.4.1 $\text{Tr}'_W(\mu_W) \neq 0$

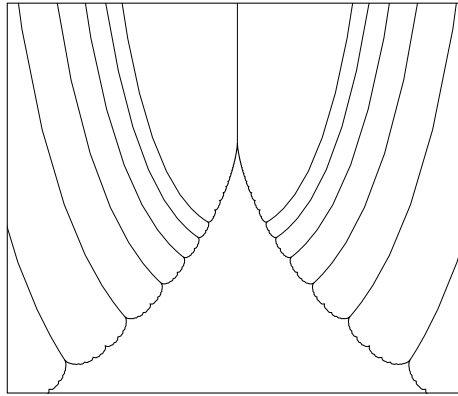


Figure 3.9: A cusp with some neighbouring cusps in the Maskit slice. The thick line is the boundary, the thinner lines are the pleating rays.

Chapter 4

Spirals in the boundary

In this chapter we define the terms of the following two theorems and prove them.

Theorem 4.0.2 *The boundaries of the Maskit and Bers slices spiral to an indefinite extent in both directions near every point.*

Theorem 4.0.3 *The boundaries of the Maskit and Bers slices spiral infinitely at an uncountable dense set of points.*

Figure 4.1 is a series of zooms into the Maskit slice. This figure gives an intuitive view of the proof of theorem 4.0.2. Computer plots suggest that if r/s is a very close Farey neighbour of p/q then the direction of the r/s -cusp is approximately perpendicular to that of the p/q -cusp (see figures 3.9 and 4.3). By starting at any cusp and repeatedly looking at very close Farey neighbours on the same side, it would seem that for any n we can find a curve within the slice which spirals at least n times around a point very close to the initial cusp in the boundary. This in turn would show that the boundary itself spirals to an indefinite extent arbitrarily near every point on the boundary. Furthermore, it would seem that at the limit point of a sequence of cusps about which the boundary spirals ever more, the boundary would spiral infinitely.

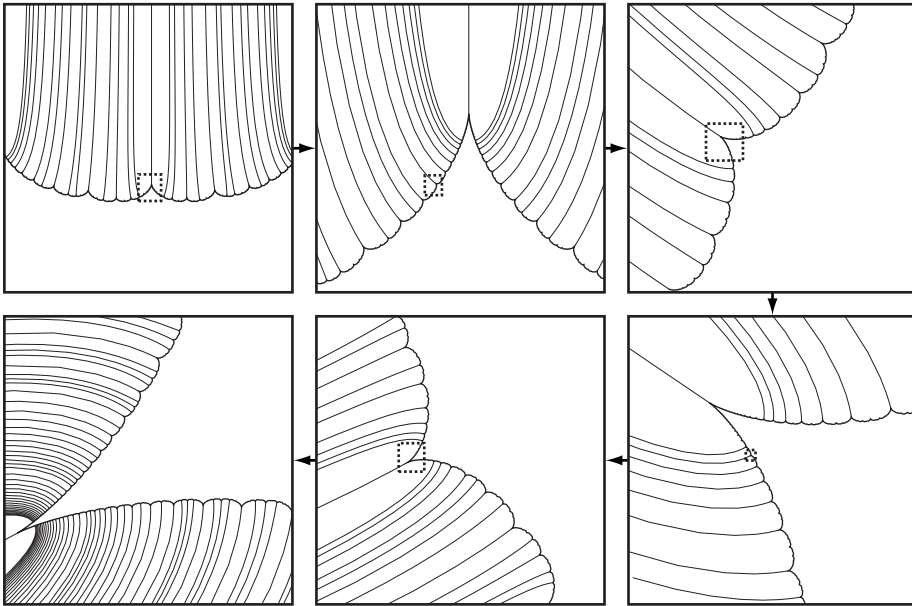


Figure 4.1: Successive zooms into the Maskit slice suggesting spiralling. The figures go from the top left to the bottom left in a clockwise order. The thick black is the boundary, the thinner lines are the pleating rays and the dashed rectangle shows the area being zoomed to in the next frame.

4.1 Spiralling

Defining the term spiralling is just a matter of capturing what is presumably already a shared intuitive idea. In figure 4.2, it is clear that both the inside and the outside of the slice “spiral around” the cusp point; and that the boundary, or any curve contained entirely inside our entirely outside the slice, “spirals around” the cusp point as well.

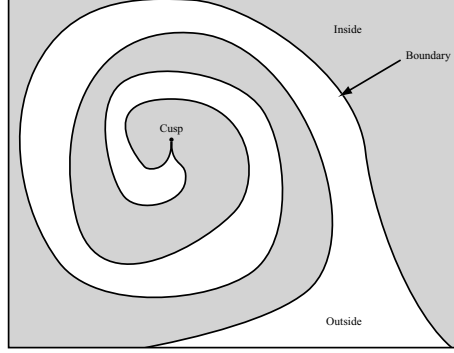


Figure 4.2: Spiralling - the outside spirals into the cusp, and so does the inside

Our definition of spiralling will be relative to a base point z_0 whose choice will turn out to be irrelevant. Suppose that $K \subseteq \mathbb{C} \cup \{\infty\}$ is any domain with ∂K a simple curve. In particular, K must be simply connected and connected. Fix the base point $z_0 \in K$. Now choose a point $z_1 \in \partial K$. Since $\overline{K} - \{z_1\}$ is simply connected and does not contain z_1 , we can define a branch of $\log(z - z_1)$ on this region. In particular, choose the unique branch with $\text{Im} \log(z_0 - z_1) \in [-\pi, \pi)$. Write $L(z) = \log(z - z_1)$.

Definition 4.1.1 *The degree of spiralling of any continuous curve $\alpha : [0, 1] \rightarrow \overline{K}$ connecting z_0 to z_1 with a smooth endpoint and $\alpha'(1) \neq 0$ is*

$$\text{sp. deg } \alpha = \lim_{t \rightarrow 1} \text{Im } L(\alpha(t)) - \text{Im } L(z_0).$$

The smoothness of α at the endpoint guarantees that $\text{sp. deg } \alpha < \infty$. Such curves may not exist (in particular, if the boundary is spiralling infinitely at that point, see below).

Definition 4.1.2 *The degree of spiralling at a boundary point z_1 is defined to be the set*

$$\text{Sp. Deg } z_1 = \{ \text{sp. deg } \alpha : \alpha \text{ joins } z_0 \text{ to } z_1 \}.$$

It turns out that for z_1 a cusp, all curves ending at z_1 have approximately the same spiralling degree, and that the exact value of the spiralling degree of a curve α depends only on the direction of the curve at the endpoint, $\alpha'(1)$. The following lemma is not necessary, but helps to justify the usefulness of the definition of spiralling given here.

Lemma 4.1.1 *If K is the Maskit or Bers slice, and $\mu_{p/q} \in \partial K$ is a cusp, then $\text{diam Sp. Deg } \mu_{p/q} \leq 2\pi$. Moreover, two curves α_1 and α_2 with smooth endpoints joining the base point μ_0 to $\mu_{p/q}$ satisfy $\alpha'_1(1)/|\alpha'_1(1)| = \alpha'_2(1)/|\alpha'_2(1)|$ if and only if either $\text{sp. deg } \alpha_1 = \text{sp. deg } \alpha_2$ or $|\text{sp. deg } \alpha_1 - \text{sp. deg } \alpha_2| = 2\pi$. Assuming $\text{sp. deg } \alpha_1 < \text{sp. deg } \alpha_2$, the latter situation can only occur if $\text{Sp. Deg } \mu_{p/q} = [\text{sp. deg } \alpha_1, \text{sp. deg } \alpha_2]$.*

Proof Let $f : \mathbb{C} \rightarrow \mathcal{R}_p(\Gamma)$ be a Maskit or Bers slice with $f(K) \subseteq \overline{\mathcal{QF}}$ as defined in section 3.2. We are considering the boundary point $\mu_{p/q}$, the p/q -cusp of a Maskit or Bers slice. The associated p/q -word is $W_{p/q} \in \Gamma$. Write $\text{Tr}_{p/q} : U \rightarrow \mathbb{C}$ for the associated trace function, defined by $\text{Tr}_{p/q}(\mu) = \text{Tr}(f(\mu)(W_{p/q}))$, chosen in a neighbourhood $U \subseteq \mathbb{C}$ of $\mu_{p/q}$ so that $\text{Tr}_{p/q}(\mu_{p/q}) = 2$. The base point is $\mu_0 \in K$ and the function L is the branch of $\log(\mu - \mu_0)$ defined on $\overline{K}$. Proposition 3.4.1 asserts

that $\text{Tr}'_{p/q}(\mu_{p/q}) \neq 0$. From this we can deduce that there is a smooth curve in the complement of $\bar{K}$ ending at the point $\mu_{p/q}$. Consider the set of points $\mu \in U$ satisfying $\text{Tr}_{p/q}(\mu) \in (0, 2)$. Let $\wp_-$ be the connected component of this set ending at $\mu_{p/q}$. Any such $\mu \in \wp_-$ cannot be in $\bar{K}$, because for such μ the word $f(\mu)(W_{p/q})$ is elliptic and therefore the associated group $f(\mu)(\Gamma)$ could not be discrete. Moreover $\wp_-$ is a smooth curve because it is contained in the real locus of an analytic function.

Choose a sufficiently small ball B_ϵ (of radius ϵ about $\mu_{p/q}$) that $\wp_- \cap B_\epsilon$ is a simple curve. $\wp_- \cap B_\epsilon$ connects $\mu_{p/q}$ to the circle ∂B_ϵ . Write $C_\epsilon = B_\epsilon - \wp_-$. Given $0 < \eta < 2\pi$ we can choose ϵ sufficiently small that $\wp_- \cap B_\epsilon$ is contained in a sector of B_ϵ of angle η . Write $K_\epsilon = (\bar{K} - \{\mu_{p/q}\}) \cap B_\epsilon$. Since $K_\epsilon \subseteq C_\epsilon$, we have that for any two points $\mu_1, \mu_2 \in K_\epsilon$, $|\text{Im } L(\mu_1) - \text{Im } L(\mu_2)| \leq 2\pi + \eta$. Letting $\eta \rightarrow 0$ we can easily see that $|\text{sp. deg } \alpha_1 - \text{sp. deg } \alpha_2| \leq 2\pi$. Write $\hat{\alpha}_i$ for $\alpha'_i(1)/|\alpha'_i(1)|$. If the spiral degree of α_1 and α_2 is the same, then clearly $\hat{\alpha}_1 = \hat{\alpha}_2$. Write $\hat{\wp}_-$ for the tangent direction of $\wp_-$ at $\mu_{p/q}$ (which is well defined because $\text{Tr}'_{p/q}(\mu_{p/q}) \neq 0$). If $\text{sp. deg } \alpha_2 = \text{sp. deg } \alpha_1 + 2\pi$ then $\hat{\alpha}_1 = \hat{\alpha}_2 = \hat{\wp}_-$. This situation occurs if the two curves α_1 and α_2 approach $\mu_{p/q}$ on different sides of $\wp_-$. It is also clear that if $0 < |\text{sp. deg } \alpha_1 - \text{sp. deg } \alpha_2| < 2\pi$ then $\hat{\alpha}_1 \neq \hat{\alpha}_2$. $\square$

Definition 4.1.3 *We say that the boundary spirals to an indefinite extent near every point if for any $n \in \mathbb{N}$ and any open neighbourhood U of any point z_1 in the boundary, there is a point $z_2 \in U \cap \partial K$ and a curve α_2 joining z_0 to z_2 such that $|\text{sp. deg } \alpha_2| > n$. We say that the boundary spirals to an indefinite extent in both directions near every point if there are two points $z_2, z_3 \in U \cap \partial K$ and two curves α_2 and α_3 joining z_0 to z_2 and z_3 respectively such that $\text{sp. deg } \alpha_2 > n$ and $\text{sp. deg } \alpha_3 < -n$.*

We extend the definition to infinite spiralling as follows (cf. [Pomm92]).

Definition 4.1.4 *The boundary spirals infinitely at z_1 if $\text{Im } L(z)$ is unbounded in every neighbourhood of z_1 .*

4.2 Cusp structure

The first stage in the proof of theorem 4.0.2 is to prove some facts about the local structure of cusp points.

Definition 4.2.1 *At a given cusp μ_∞ with associated trace function Tr_∞ chosen near μ_∞ so that $\text{Tr}_\infty(\mu_\infty) = 2$, the sequence μ_n^* is defined as follows.*

$$\mu_n^* = \mu_\infty - \frac{\pi^2}{n^2 \text{Tr}'_\infty(\mu_\infty)}$$

Any sequence $\tilde{\mu}_n$ satisfying $\tilde{\mu}_n = \mu_n^ + O(n^{-3})$ is said to be an approximate cusp sequence.*

In particular, we will prove:

Lemma 4.2.1 *Let μ_n be the sequence of cusps which are neighbours of μ_∞ as in section 3.3, then the sequence μ_n is an approximate cusp sequence. That is, $\mu_n = \mu_\infty - \frac{\pi^2}{n^2 \text{Tr}'_\infty(\mu_\infty)} + O(n^{-3})$.*

The method of proving this comes from [Miyachi03]. We use:

Theorem 4.2.2 (Pivot Theorem, [Minsky99]) *There exist positive constants ϵ, c_1 such that, if $\rho: \Gamma \rightarrow \text{PSL}_2\mathbb{C}$ is a marked punctured-torus group and $\ell(\alpha) \leq \epsilon$ then*

$$\frac{2\pi i}{\lambda(\alpha)} \approx \nu_+(\alpha) - \overline{\nu_-(\alpha)} + i$$

where “ $\approx$ ” denotes a bound c_1 on hyperbolic distance in $\mathbb{H}^2$ between the left and right sides.

Here α is an element of the fixed group $\Gamma = \langle X, Y \rangle$, $\lambda(g)$ is the complex length of the element $g \in \mathrm{PSL}_2\mathbb{C}$ defined by the equation $\mathrm{Tr}(g) = 2 \cosh \lambda(g)/2$, $\lambda(\alpha) = \lambda(\rho(\alpha))$, $\ell(\alpha) = \mathrm{Re} \lambda(\rho(\alpha))$ and $\nu_{\pm}(\alpha)$ are the normalised end invariants of the two boundary components of the associated hyperbolic 3-manifold. The normalisation is to choose an element $T \in \mathrm{PSL}_2\mathbb{Z}$ sending p/q to ∞ where α is the p/q -word in Γ , and to set $\nu_{\pm}(\alpha) = T(\nu_{\pm})$ where $\nu_{\pm}$ are the end invariants or Teichmüller parameters of ρ . It is important to note that we think of $\ell(\alpha)$, $\nu_{\pm}(\alpha)$ and $\lambda(\alpha)$ as functions on $\overline{\mathcal{QF}}$. In [Minsky99], Minsky is considering a fixed representation whereas we are considering a varying representation. So for example, we define:

$$\ell(\alpha) : \overline{\mathcal{QF}} \rightarrow \mathbb{R}; [\rho] \rightarrow \mathrm{Re} \lambda(\rho(\alpha)).$$

We are considering a slice embedded in $\mathbb{C}$ via a holomorphic map $f : \mathbb{C} \rightarrow \mathcal{R}_p(\Gamma)$, so we can write $[\rho] = f(\mu)$ as a holomorphic function of a complex parameter μ . The cusp $[\rho_{\infty}]$ (corresponding to pinching α) corresponds to the point μ_{∞} . With this understood, we can write

$$\mathrm{Tr}(\rho(\alpha)) = 2 + (\mu - \mu_{\infty}) \mathrm{Tr}'_{\infty}(\mu_{\infty}) + O((\mu - \mu_{\infty})^2). \quad (4.1)$$

So we can either think of $\mathrm{Tr}(\rho(\alpha))$ as itself a local coordinate, or we can use this Taylor expansion to find the μ coordinate. In this section we have chosen the sign of the trace function near $[\rho_{\infty}]$ so that $\mathrm{Tr}(\rho_{\infty}(\alpha)) = 2$.

The method is as follows. We wish to estimate the value of μ giving rise to a representation ρ near ρ_{∞} and in the boundary of the slice. Using the fact that Tr is holomorphic and the derivative is nonzero at μ_{∞} (by proposition 3.4.1), it suffices to estimate $\mathrm{Tr}(\rho(\alpha))$. Using the formula $\mathrm{Tr}(g) = 2 \cosh \lambda(g)/2$, it suffices to estimate $\lambda(\rho(\alpha))$. We identify the normalised end invariants $\nu_{\pm}(\alpha)$, and use the pivot theorem to estimate $\lambda(\rho(\alpha))$ (subsequently, just written λ). This gives us an estimate $2\pi i/\lambda_0 = \omega_0$ of $2\pi i/\lambda = \omega$, which in turn gives us an estimate λ_0 for λ .

Proof of lemma 4.2.1 The boundary of the Maskit or Bers slices is a homeomorphic image of $\mathbb{R}$ or $\hat{\mathbb{R}}$, so that in a neighbourhood of the α cusp, $\ell(\alpha) \leq \epsilon$ (as ℓ is continuous) and the pivot theorem applies. Writing $B_r(z)$ for the hyperbolic ball of radius r about the point $z \in \mathbb{H}$, we have in this case that:

$$\omega = 2\pi i/\lambda \in B_{c_1}(\nu_+(\alpha) - \overline{\nu_-(\alpha)} + i).$$

Define the point $\omega_0 = \nu_+(\alpha) - \overline{\nu_-(\alpha)} + i$ (in the upper half plane).

Let $T \in \mathrm{PSL}_2\mathbb{Z}$ be the normalisation so that $T(p/q) = \infty$. Writing $T(p/q) = \frac{az+b}{cz+d}$, we need to find integers a, b, c, d such that $T(p/q) = \infty$ and $ad - bc = 1$. Trying $c = q, d = -p$ we get that $T(p/q) = \infty$ and we need a, b such that $ap + bq = -1$ which can be solved by Euclid's algorithm. This gives:

$$T(z) = \frac{a}{q} + \frac{1}{q^2(z - p/q)}.$$

So in this notation, $\omega_0 = T(\nu_+) - \overline{T(\nu_-)} + i$. This in turn gives $\omega_0 = q^{-2}(\nu_+ - p/q)^{-1} + c$ for a constant c . An essential point here is that since ν_- is fixed in the Bers and Maskit slices, $T(\nu_-)$ is just a constant. Only ν_+ is varying. We define the symbol τ , which will be used again later, as follows:

$$\tau = \frac{1}{q^2(\nu_+ - p/q)}.$$

With this symbol, we can write $\omega_0 = \tau + c$. Write $d_{\mathbb{H}}$ for the hyperbolic metric on the upper half plane, and $d_{i\mathbb{H}}$ for the hyperbolic metric on the right hand half plane, then $\mathbb{H} \rightarrow i\mathbb{H}; z \mapsto 2\pi i/z$ is an isometry.

Now define λ_0 by $\omega_0 = 2\pi i/\lambda_0$. The pivot theorem says that $d_{\mathbb{H}}(\omega, \omega_0) \leq c_1$, and so using the isometry we get that $d_{i\mathbb{H}}(\lambda, \lambda_0) \leq c_1$. The hyperbolic ball of radius c_1 centred at λ_0 in $i\mathbb{H}$ has Euclidean radius $\mathrm{Re} \lambda_0 \cdot \sinh c_1$, so we get the bound $|\lambda - \lambda_0| \leq 2 \mathrm{Re} \lambda_0 \cdot \sinh c_1$ for the Euclidean distance.

Now $\lambda_0 = 2\pi i/\omega_0 = 2\pi i/(\tau + c) = 2\pi i/\tau + O(\tau^{-2})$. So $\mathrm{Re} \lambda_0 = O(\tau^{-2})$ and hence $d_{\mathbb{E}}(\lambda, \lambda_0) = O(\tau^{-2})$. Hence $\lambda = 2\pi i/\tau + O(\tau^{-2})$. Using $\mathrm{Tr} W = 2 \cosh \lambda(W)/2$ and the Taylor series for $\cosh$ we get that $\mathrm{Tr} \rho \alpha = 2 - \frac{\pi^2}{\tau^2} + O(\tau^{-3})$. Equating this with equation 4.1, we get that $\mu - \mu_{\infty} =$

$$-\frac{\pi^2}{\tau^2 \operatorname{Tr}'_\infty(\mu_\infty)} + O(\tau^{-3}) + O((\mu - \mu_\infty)^2).$$

Explicitly, write $\operatorname{Tr} \rho\alpha = 2 - \frac{\pi^2}{\tau^2} + A_\tau$, where $|A_\tau| \leq A|\tau|^{-3}$ (for some constant A). Using equation 4.1 we write $\operatorname{Tr} \rho\alpha = 2 + (\mu - \mu_\infty) \operatorname{Tr}'_\infty(\mu_\infty) + B_\mu$ where $|B_\mu| \leq B|\mu - \mu_\infty|^2$ (some B). Equating we get that $\mu - \mu_\infty = -\frac{\pi^2}{\tau^2 \operatorname{Tr}'_\infty(\mu_\infty)} + A_\tau / \operatorname{Tr}'_\infty(\mu_\infty) - B_\mu / \operatorname{Tr}'_\infty(\mu_\infty)$. Write $C_\mu = \mu - \mu_\infty + B_\mu / \operatorname{Tr}'_\infty(\mu_\infty)$ and $C_\tau = -\frac{\pi^2}{\tau^2 \operatorname{Tr}'_\infty(\mu_\infty)} + A_\tau / \operatorname{Tr}'_\infty(\mu_\infty)$. Clearly $C_\mu = C_\tau$, and $C_\tau = O(\tau^{-2})$ so we can write $|C_\tau| \leq C|\tau|^{-2}$ (for some C). It is also clear that for small $|\mu - \mu_\infty|$ we have $|C_\mu| \geq D|\mu - \mu_\infty|$ (for some D). Putting these together we get that $D|\mu - \mu_\infty| \leq C|\tau|^{-2}$. Squaring this, we get that $|B_\mu| \leq BC^2|\tau|^{-4}/D^2$. So $A_\tau - B_\mu = O(\tau^{-3})$. In conclusion:

$$\mu = \mu_\infty - \frac{\pi^2}{\tau^2 \operatorname{Tr}'_\infty(\mu_\infty)} + O(\tau^{-3}).$$

Finally, suppose p/q and r/s are Farey neighbours, and define $r_n = \frac{np+r}{nq+s}$, $r_\infty = p/q$. Then $r_n \rightarrow r_\infty$ and r_n is a Farey neighbour of r_∞ . In fact, the r_n are the fractions corresponding to the sequence of neighbouring cusps visible in figures 3.9 and 4.3. If $\nu_+ = r_n$ then $\tau = \pm(n + s/q)$ depending on whether $ps - rq = \mp 1$. This gives us the estimate

$$\mu_n = \mu_\infty - \frac{\pi^2}{n^2 \operatorname{Tr}'_\infty(\mu_\infty)} + O(n^{-3}),$$

as required. $\square$

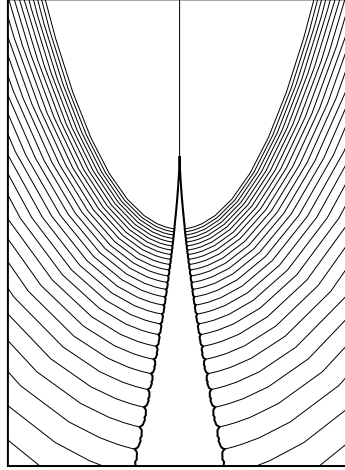


Figure 4.3: A close up view of a cusp with some neighbouring cusps in the Maskit slice. The thick line is the boundary, the thinner lines are the pleating rays.

In fact, we can actually say, by looking at Miyachi's argument in [Miyachi03] (in particular, the proof of proposition 4.3), that we must have $a\tau^{-2} < \operatorname{Re} \lambda < b\tau^{-2}$ for $0 < a < b$ and $\operatorname{Im} \lambda = 2\pi/\tau + O(\tau^{-2})$. Similarly, we can say that $\operatorname{Re} \operatorname{Tr} \rho\alpha = 2 - \pi^2/\tau^2 + O(\tau^{-3})$, and that there are constants $0 < a < b$ such that $a\tau^{-3} < |\operatorname{Im} \operatorname{Tr} \rho\alpha| < b\tau^{-3}$. This gives the cusps the local structure discussed in section 3.3. This refinement will be important later on in section 4.4.

4.3 Trace derivatives

Suppose now that $\Gamma = \langle X, Y \rangle$ is a fixed free group on two generators X and Y , and that Γ is generated by A and B so that in the boundary of the slice the cusps corresponding to A and B are neighbours. Let $f : \mathbb{C} \rightarrow \mathcal{R}_p(\Gamma)$ be a slice as in section 3.2, and let A and B be the p/q and r/s words with respect to the generators X and Y . Assume that $p/q < r/s$. Defining $r_n = (np+r)/(nq+s)$, we get that the r_n word is $A^n B$. This is the sequence of neighbouring cusps between the A and B cusps (see figure 4.4 and compare with figure 3.7). If we had that $r/s < p/q$ then this sequence of neighbouring cusps would be BA^n and the conclusions would be mostly but not exactly the same,

the points at which the conclusions are different will be highlighted. As before, $\text{Tr}_n = \text{Tr}_{A^n B}$ and $\text{Tr}_\infty = \text{Tr}_A$. Below, we will make a careful choice of the signs of these functions.

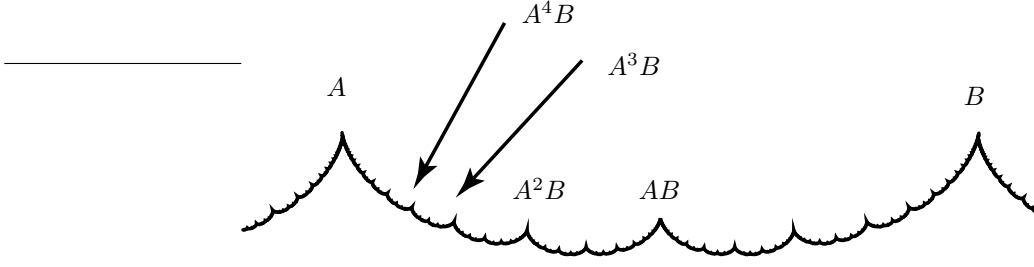


Figure 4.4: Enumerating neighbouring cusps

Lemma 4.3.1 *If a sequence of points $\tilde{\mu}_n$ is an approximate cusp sequence, then*

$$\text{Tr}'_n(\tilde{\mu}_n) = \frac{\pm n^3 i \text{Tr}'_\infty(\mu_\infty)}{\pi^2} + O(n^2).$$

In particular, since the sequence of neighbouring cusp points μ_n is an approximate cusp sequence, then by lemma 4.2.1 we have

Corollary 4.3.2 *For the sequence of neighbouring cusps,*

$$\text{Tr}'_n(\mu_n) = \frac{\pm n^3 i \text{Tr}'_\infty(\mu_\infty)}{\pi^2} + O(n^2).$$

In particular, the initial directions of two close Farey neighbouring pleating rays are approximately perpendicular to one another.

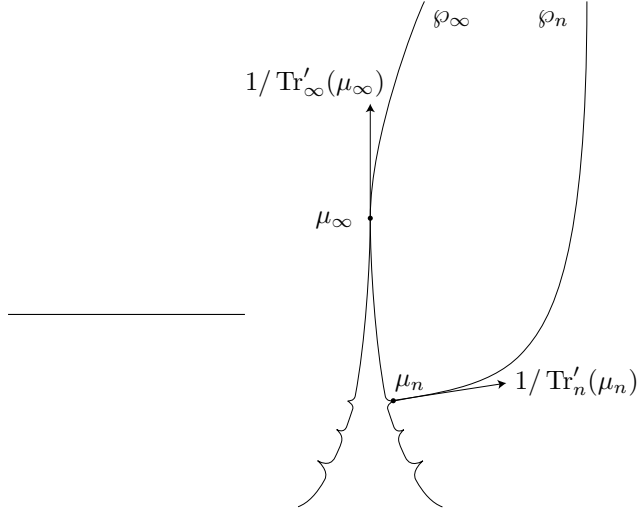


Figure 4.5: Schematic view of perpendicularity phenomenon

Proof of lemma 4.3.1 The Markov identity states that for elements $A, B \in \text{SL}_2\mathbb{C}$ we have that

$$(\text{Tr } A)^2 + (\text{Tr } B)^2 + (\text{Tr } AB)^2 = \text{Tr } A \cdot \text{Tr } B \cdot \text{Tr } AB.$$

Choose $\text{Tr}_A = \text{Tr}_\infty$ near μ_∞ so that $\text{Tr}_\infty \mu_\infty = 2$. Now choose a sign for Tr_B , and this will determine a sign choice for Tr_{AB} so that Markov's identity holds in a neighbourhood of μ_∞ . We

use the following formula, easily proved by induction for $A, B \in \text{SL}_2\mathbb{C}$,

$$\text{Tr } A^n B = ae^{n\lambda/2} + be^{-n\lambda/2}, \quad (4.2)$$

where a and b are constants depending on A and B , and λ is the complex length of A defined by $\text{Tr } A = 2 \cosh \lambda/2$. (Note that if we were dealing with the case $r/s < p/q$ this formula would be the same because $\text{Tr } A^n B = \text{Tr } B A^n$.) If $n = 0$ we get $a + b = \text{Tr}_B$ and if $n = 1$ we get $ae^{\lambda/2} + be^{-\lambda/2} = \text{Tr}_{AB}$. Solving these gives

$$a = \frac{\text{Tr}_{AB} e^{\lambda/2} - \text{Tr}_B}{e^{\lambda} - 1}, \quad b = \frac{e^{\lambda} \text{Tr}_B - e^{\lambda/2} \text{Tr}_{AB}}{e^{\lambda} - 1}. \quad (4.3)$$

With the choice of functions Tr_A , Tr_B and Tr_{AB} already made, define a choice of sign for Tr_n using the formula $\text{Tr}_n = ae^{n\lambda/2} + be^{-n\lambda/2}$. It may or may not be the case with this choice that $\text{Tr}_n(\mu_n) = 2$. The sign ambiguity this introduces will be resolved in lemma 4.4.1. Also note that since Tr_{AB} , Tr_B and λ are holomorphic functions of μ , so are a and b .

Let $\tilde{\mu}_n$ be an approximate cusp sequence, that is

$$\tilde{\mu}_n = \mu_\infty - \frac{\pi^2}{n^2 \text{Tr}'_\infty(\mu_\infty)} + O(n^{-3}). \quad (4.4)$$

We know that

$$\text{Tr}_\infty(\mu) = 2 + (\mu - \mu_\infty)T + O(\mu - \mu_\infty)^2 \quad (4.5)$$

where T is defined as $\text{Tr}'_\infty(\mu_\infty) \in \mathbb{C}$. Thinking of λ as a function in μ , so that

$$\lambda(\mu) = 2 \cosh^{-1} \text{Tr}_\infty(\mu)/2, \quad (4.6)$$

we have

$$\lambda(\mu) = 2\sqrt{(\mu - \mu_\infty)T} + O(\mu - \mu_\infty)^{3/2}. \quad (4.7)$$

The square roots in the expression above are not a problem. We will only differentiate λ at $\tilde{\mu}_n \neq \mu_\infty$ and the sign ambiguity that is introduced by the use of the square root will be resolved in lemma 4.4.1.

We start by differentiating equation 4.6 and substituting equation 4.5 to get

$$\lambda'(\mu) = \frac{(\text{Tr}_\infty)'(\mu)}{\sqrt{T(\mu - \mu_\infty) + O(\mu - \mu_\infty)^2}}. \quad (4.8)$$

Substituting equation 4.4 for $\tilde{\mu}_n$ in equation 4.7 we get

$$\lambda(\tilde{\mu}_n) = \pm \frac{2i\pi}{n} + O(n^{-3/2}). \quad (4.9)$$

Similarly we get

$$\lambda'(\tilde{\mu}_n) = \mp \frac{inT_n}{\pi} + O(1) \quad (4.10)$$

where $T_n = \text{Tr}'_\infty(\tilde{\mu}_n)$. Since Tr_∞ is holomorphic at μ_∞ and $\tilde{\mu}_n \rightarrow \mu_\infty$ we have that $T_n \rightarrow T$. In fact, $T_n = T + O(\tilde{\mu}_n - \mu_\infty) = T + O(n^{-2})$. Finally, we differentiate equation 4.2 to get

$$\text{Tr}'_n = a'e^{n\lambda/2} + b'e^{-n\lambda/2} + an\lambda'/2 \cdot e^{n\lambda/2} - bn\lambda'/2 \cdot e^{-n\lambda/2}. \quad (4.11)$$

Evaluating this at $\tilde{\mu}_n$ and using all of the facts above gives, after some simplification, that

$$\text{Tr}'_n(\tilde{\mu}_n) = \pm \frac{(\text{Tr}_{AB} - \text{Tr}_B)n^3T}{2\pi^2} + O(n^2). \quad (4.12)$$

We now apply Markov's identity. Since $\text{Tr}_A(\tilde{\mu}_n) = \text{Tr}_\infty(\tilde{\mu}_n) = 2 + O(n^{-2})$ we get $4 + (\text{Tr}_B)^2 + (\text{Tr}_{AB})^2 = 2\text{Tr}_B \cdot \text{Tr}_{AB} + O(n^{-2})$ which can be factorised to give $(\text{Tr}_B - \text{Tr}_{AB})^2 = -4 + O(n^{-2})$ or

$\text{Tr}_B - \text{Tr}_{AB} = \pm 2i + O(1/n)$. Substituting this in equation 4.12 gives us the required result. $\square$

4.4 Curve construction

At this point, we almost have theorem 4.0.2. It would seem as if we could take a cusp, find a close neighbouring cusp approximately perpendicular to it, do the same to this cusp and repeat again and again, each time rotating our view by 90 degrees until we get as much spiralling as we like. There are two problems. First of all, there is a sign ambiguity in $\text{Tr}'_n(\tilde{\mu}_n)$ in lemma 4.3.1. This might mean that our choice of neighbours might first increase the angle by 90 degrees, and then decrease it by 90 degrees, and so on so that we end up with no spiralling at all. The other possibility is illustrated in figure 4.6. In this case, we would actually rotate the angle by -270 degrees rather than +90 degrees.

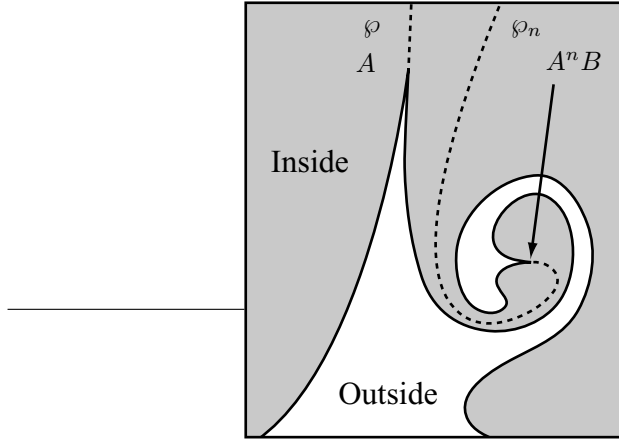


Figure 4.6: A logically possible monstrosity

In [Wright88], David Wright conjectured that the straight line segment between any two neighbouring cusps in the Maskit slice is entirely contained within the slice. This is the conjecture referred to at the beginning of section 3.3. This conjecture would rule out the possibility of something like figure 4.6. Below, we prove something almost as good as Wright's conjecture, at least for the purposes of this paper, that there is an arc between two close neighbouring cusps consisting of an almost straight segment of length $O(n^{-3})$ in the direction of the $A^n B$ cusp followed by an almost straight segment of length $O(n^{-2})$ in the direction of the A cusp. This is also good enough to rule out the possibility of figure 4.6.

The construction of this arc is roughly as follows, illustrated in figure 4.7 (the $O(-)$ notation refers to the length of the two components of the arc). Near a close neighbouring cusp, the trace derivative satisfies the perpendicularity equation of lemma 4.3.1. Suppose $\text{Tr}_n(\mu_n) = 2$, and the pleating ray is $\wp_n(t)$ parameterised so that $\text{Tr}_n(\wp_n(t)) = t$, then the direction of the pleating ray at $\wp_n(t)$ is $1/\text{Tr}'_n(\wp_n(t))$ by a simple application of the chain rule. So, at those points $\wp_n(t)$ within some small neighbourhood of the $A^n B$ -cusp, the direction of the pleating ray will be approximately perpendicular to the direction of the pleating ray of the main cusp. It turns out that the size of the neighbourhood of the $A^n B$ -cusp in which this is true is large enough that the pleating ray reaches one of the bounding $(2, 3)$ -cusps of the main cusp. The arc is then the initial section of the pleating ray $\wp_n$ to the point where it touches one of the bounding $(2, 3)$ -cusps, followed by the segment of the bounding $(2, 3)$ -cusp from the intersection point to the main cusp. More explicitly, let ψ_+ be the outer bounding $(2, 3)$ -cusp curve. The arc we construct consists of the initial segment of $\wp_n$ (from $\wp_n(2)$ to $\wp_n(t_0)$ where $\wp_n(t_0)$ is the first point of intersection of $\wp_n$ and ψ_+), followed by the segment of ψ_+ from this intersection point to the main cusp.

Lemma 4.3.1 says that for any approximate cusp sequence $\tilde{\mu}_n$, we get an equation for $\text{Tr}'_n(\tilde{\mu}_n)$ depending only on n with an $O(n^2)$ term. Here we are not assuming that $\tilde{\mu}_n$ are the neighbouring cusps. Instead, suppose that μ_n is the sequence of cusp neighbours of μ_∞ and $\tilde{\mu}_n$ is any sequence with $\tilde{\mu}_n = \mu_n + O(n^{-3})$ then $\tilde{\mu}_n$ will be an approximate cusp sequence and lemma 4.3.1 will apply.

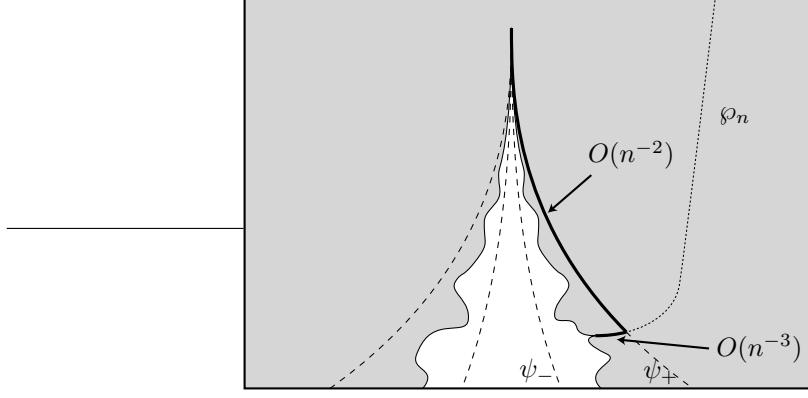


Figure 4.7: Constructing an almost straight arc between two neighbouring cusps

For ease of discussion, rotate and translate the picture so that the main cusp direction is straight upwards and the main cusp is at 0. Without loss of generality, we consider only the right hand side of the picture. Figure 4.8 shows this. We write $\phi(\tau)$ for the parameterisation of the boundary and $\psi_+(\tau)$ for the outer bounding (2, 3)-cusp. Here $\tau \in \mathbb{R}$ and $\phi(\tau)$ is the point on the boundary such that $\nu_+ = p/q + q^2\tau$ (see the definition of τ in the proof of lemma 4.2.1, section 4.2). Choosing the constants appropriately, we say that $\psi_+(\tau) = -ai\tau^{-2} + b\tau^{-3}$ (see ψ_+ marked on figure 4.7). It is easy to see from the fact that we have an $O(\tau^{-3})$ estimate of $\phi(\tau)$ that the horizontal distance from $\phi(\tau)$ to the outer bounding curve ψ_+ is $O(\tau^{-3})$. Say this distance is less than $W\tau^{-3}$.

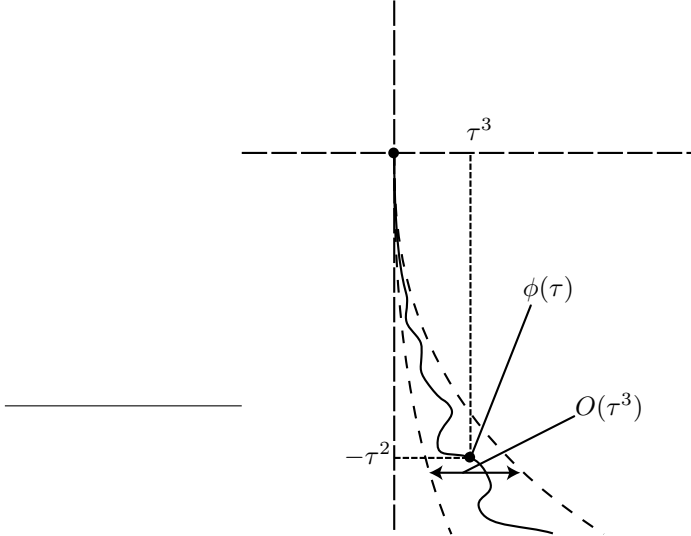


Figure 4.8: Further local structure of the boundary near a cusp

Before proceeding with the construction of the arc, we slightly improve lemma 4.3.1. The proof of this lemma gives the idea of the construction of the arc.

Lemma 4.4.1 *Let $\tilde{\mu}_n$ be an approximate cusp sequence, and let $\text{Tr}_n(\mu_n) = 2\epsilon_n$ (so $\epsilon_n = \pm 1$). Then*

$$\text{Tr}'_n(\tilde{\mu}_n) = \frac{\epsilon_n n^3 i \text{Tr}'_\infty(\mu_\infty)}{\pi^2} + O(n^2).$$

Notice that if we had assumed that $r/s < p/q$ then we would need to prove $\text{Tr}'_n(\tilde{\mu}_n) = \frac{-\epsilon_n n^3 i \text{Tr}'_\infty(\mu_\infty)}{\pi^2} + O(n^2)$. The point is that if the cusp neighbours are on the right hand side of the main cusp then they point out one way, and if they are on the left they point out the other way (either way, they point away from the axis of the main cusp).

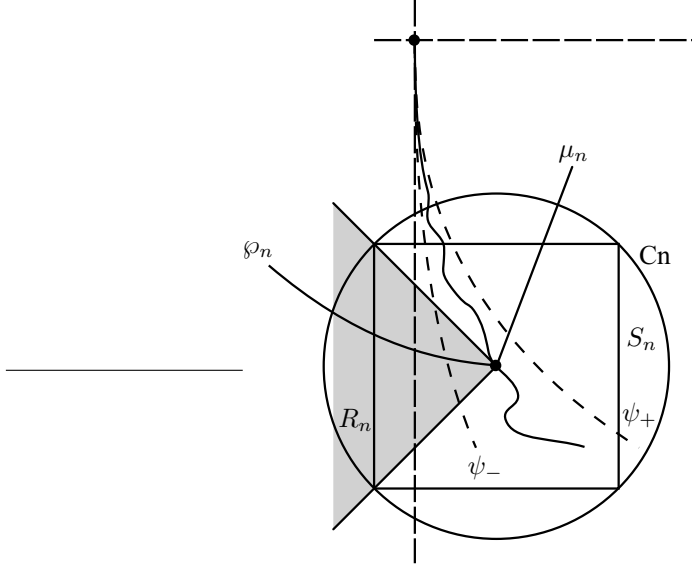


Figure 4.9: Impossible consequence of $\delta = -1$ in equation 4.13

Proof Without loss of generality, suppose $\epsilon_n = 1$. In this case the initial direction of the pleating ray is $1/\text{Tr}'_n(\mu_n)$. If $\epsilon_n = -1$ then it would be $-1/\text{Tr}'_n(\mu_n)$. Let

$$\text{Tr}'_n(\tilde{\mu}_n) = \frac{\delta n^3 i \text{Tr}'_\infty(\mu_\infty)}{\pi^2} + O(n^2) \quad (4.13)$$

for $\delta = \pm 1$. Suppose we have that $\delta = -1$. This leads to the contradiction, illustrated in figure 4.9, that the pleating ray leaves the slice.

We can certainly say that $|\text{Im } \phi(\tau)| \leq W\tau^{-3}$ for some constant W . Define C_τ to be the disc of radius $\sqrt{2}W\tau^{-3}$ about $\phi(\tau)$ so that the inscribed square S_τ intersects the imaginary axis. Now, for any point $\tilde{\mu}_n \in C_n$ we have that $\tilde{\mu}_n = \mu_n + O(n^{-3})$ because the radius of C_n is $O(n^{-3})$, and so any sequence of $\tilde{\mu}_n \in C_n$ is an approximate cusp sequence. Here C_n is defined to be C_τ for the value of τ corresponding to n , the precise value is $\tau = n + s/q$ (see the definition of τ in the proof of lemma 4.2.1). Lemma 4.3.1 then says that equation 4.13 will hold for any $\tilde{\mu}_n \in C_n$.

Choose N large enough so that for $n \geq N$ we have that the angle from the horizontal of $\phi'_n = 1/\text{Tr}'_n(\tilde{\mu}_n)$ is less than $\pi/4$. Let R_n be the conical region of angle $\pi/4$ with the horizontal, centered at the $A^n B$ cusp μ_n , expanding in the direction of the negative horizontal axis. Now, $\phi'_n(t) = 1/\text{Tr}'_n(\phi_n(t))$. Since $1/\text{Tr}'_n(\phi_n(s))$ has angle less than $\pi/4$ to the horizontal as long as $\phi_n(s) \in R_n$, the intersection of ϕ_n with C_n is contained within R_n (if ϕ_n left the region R_n it would have a tangent with angle greater than $\pi/4$ with the horizontal at that point). In fact, it is possible that ϕ_n could leave the region R_n and subsequently come back, but we need only consider the initial segment of ϕ_n . More precisely then, the connected component of the intersection of ϕ_n with C_n containing the cusp μ_n is contained within R_n . In particular, since S_n intersects the negative imaginary axis so must ϕ_n (otherwise it would remain within a bounded region of the cusp). However, points on the negative imaginary axis are never inside the slice, whereas points on ϕ_n are always within the slice. This contradiction tells us that $\delta = 1$ in equation 4.13. $\square$

The construction of the curve from the $A^n B$ cusp to the main cusp works in much the same way. However, rather than joining the $A^n B$ cusp to the imaginary axis via the initial segment of the pleating ray, we join it to the outer bounding $(2, 3)$ -cusp via the pleating ray, and from there to the main cusp via the outer bounding $(2, 3)$ -cusp. Consider figure 4.10. The conical region R_τ shown shaded is defined to be centred at $\phi(\tau)$ and bounded by two lines with gradient $\pm m_1$ where m_1 is any positive number. R_τ is expanding in the direction of the positive horizontal axis. We want to show that:

Lemma 4.4.2 *For τ large enough, and for some constant D , the conical region R_τ always intersects the outer bounding curve ψ_+ within a distance $D\tau^{-3}$ of $\phi(\tau)$.*

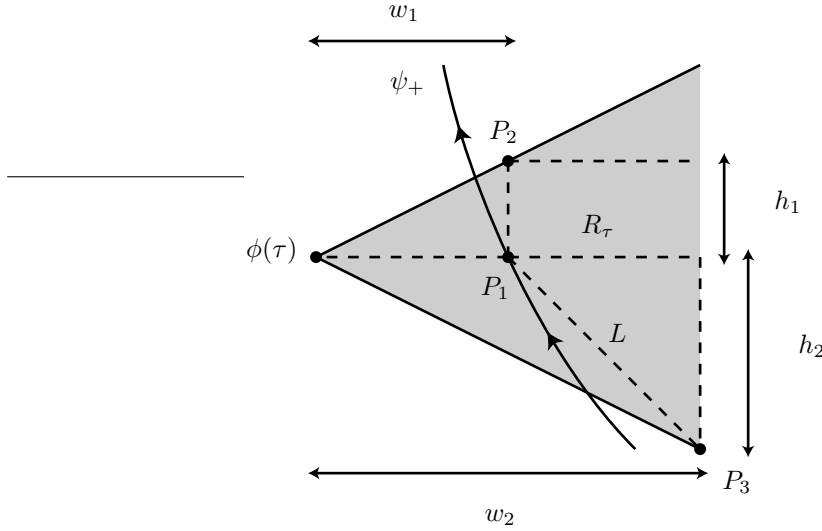


Figure 4.10: Convexity argument

Proof We know that the outer bounding curve is $\psi_+(\tau) = -ai\tau^{-2} + b\tau^{-3}$ (for some $a, b \in \mathbb{R}$). It is important to note here that the parameterisations of $\psi_+(\tau)$ and $\phi(\tau)$ are both such that as τ increases to ∞ , the point gets closer and closer to μ_∞ . In figure 4.10 this is illustrated by arrows on the ψ_+ curve. In particular, keep this in mind when thinking about the gradients of tangents to this curve. Differentiating $\psi_+(\tau)$, we get that $\psi_+(\tau)' = 2ai\tau^{-3} - 3b\tau^{-4}$. This gives the gradient from the horizontal of the tangent line a value $m(\tau) = -2a\tau/3b$. Since this tends to $-\infty$ as τ increases, we can say that for $\tau > T_1$ we have $|m(\tau)| > m_2$ for a constant m_2 which we will choose to be anything larger than m_1 (we could choose $m_i = i$ for instance).

On figure 4.10 the point labelled P_1 is the intersection of ψ_+ with the horizontal line through $\phi(\tau)$. The point labelled P_2 is the point on the intersection of the vertical line through P_1 and the upper bounding line of the cone. The point labelled P_3 is the intersection of the line L , which is from P_1 with gradient $-m_2$, with the lower bounding line of the cone. We define w_1 to be the distance from $\phi(\tau)$ to P_1 , h_1 to be the distance from P_1 to P_2 , w_2 to be the distance from $\phi(\tau)$ to the vertical line through P_3 , and h_2 to be the distance from P_3 to the horizontal line through $\phi(\tau)$.

We get that $h_1/w_1 = h_2/w_2 = m_1$ and $h_2/(w_2 - w_1) = m_2$. This gives $h_1 = m_1 w \leq m_1 W \tau^{-3}$ and $h_2 = w/(1/m_1 - 1/m_2) \leq W \tau^{-3}/(1/m_1 - 1/m_2)$. We want to define a constant T_2 such that if $\tau > T_2$ and for some other value, say $\tilde{\tau}$, we have that $\text{Im } \psi_+(\tilde{\tau}) < \text{Im } \phi(\tau) - h_2$ (that is, $\psi_+(\tilde{\tau})$ is below a horizontal line through the point P_1 in figure 4.10) then we have that $\tilde{\tau} > T_1$. This can easily be arranged using the formula for ψ_+ and the estimate for $\text{Im } \phi(\tau)$. Now for any $\tau > T_2$, because $|m(\tilde{\tau})| > m_2$ whenever $\text{Im } \psi_+(\tilde{\tau}) > \text{Im } \phi(\tau) - h_2$, the upper (resp. lower) bounding line of the cone intersects ψ_+ within a distance $\sqrt{w_1^2 + h_1^2}$ (resp. $\sqrt{w_2^2 + h_2^2}$), the distance from $\phi(\tau)$ to P_2 (resp. P_3). This gives us the required constant D . $\square$

Now we proceed to construct a curve as in the proof of lemma 4.4.1. For all sequences of points $\tilde{\mu}_n \in C_n$ (the circle centred around the n th neighbouring cusp) we have that $\tilde{\mu}_n$ is an approximate cusp sequence. Choose N large enough that the absolute value of the gradient of a line in the direction $1/\text{Tr}'_n$ is smaller than m_1 for $n \geq N$. Points on $\wp_n$ within a distance D/n^3 of the n th cusp must be within the conical region R_n which intersects the outer bounding $(2,3)$ -cusp ψ_+ , so the pleating ray must also intersect ψ_+ . (As before, we are only interested in the connected component of $\wp_n$ intersected with C_n containing μ_n , this initial segment of $\wp_n$ must be contained in R_n .) This gives us the curve we wanted, just connect the initial segment of $\wp_n$ from the n th cusp to its intersection with ψ_+ to the segment of ψ_+ from this intersection point to the main cusp.

4.5 Main Theorems

Proof of theorem 4.0.2 Given a point on the boundary, connect the base point z_0 to any very close cusp z_1 by a curve α_1 as in the definition of spiralling in section 4.1. Let z_2 be a very close neighbouring cusp of z_1 on the right hand side. Construct the curve α_2 by joining to the end of α_1 the two segments described above (the initial segment of the pleating ray coming out of z_2 followed by the outer bounding (2,3)-cusp of the z_1 cusp). Let L_1 and L_2 be the branches of the log function L , in the definitions in section 4.1, corresponding to z_1 and z_2 . Now $L_1(z) - L_2(z) = \log \frac{z-z_1}{z-z_2}$. So, if we look at only those points on α_1 and α_2 which don't get too close to z_1 or z_2 (say, exclude a small η -neighbourhood of z_1 so that $|z_2 - z_1| < \eta/\kappa$ for some large κ), we can make $L_1(\alpha_1(t)) - L_2(\alpha_1(t))$ as small as we like by making η small and κ large. This is because $|\frac{z-z_1}{z-z_2} - 1| = |\frac{z_2-z_1}{z-z_2}| \leq 1/\kappa$, $\log(1) = 0$ and $\log$ is continuous at 1. Note that since we are choosing an arbitrarily close neighbour z_2 of z_1 we can ensure that z_2 is in the η -neighbourhood of z_1 that we choose. It will be important in the proof of theorem 4.0.3 that there are a countably infinite number of choices for the point z_2 , any of the countably infinitely many cusp neighbours within the η -neighbourhood of z_1 will do.

Suppose then that except in this neighbourhood of z_1 we have that $|L_1(\alpha_1(t)) - L_2(\alpha_1(t))| < \delta$ for some small δ , say $\delta = \pi/24$. Now, we know what α_1 and α_2 look like in this small neighbourhood of z_1 , and we can easily see that $\text{sp. deg } \alpha_2 \leq \text{sp. deg } \alpha_1 - \pi/3 + 2\delta$. This gives us that $\text{sp. deg } \alpha_2 \leq \text{sp. deg } \alpha_1 - \pi/4$. Repeating this procedure, we get that $\text{sp. deg } \alpha_n \leq \text{sp. deg } \alpha_1 - n\pi/4$ and in particular that $\text{sp. deg } \alpha_n \rightarrow -\infty$ as $n \rightarrow \infty$.

So the boundary spirals to an indefinite extent near every point. By choosing neighbouring cusps on the left hand side instead of the right, we can find curves spiralling arbitrarily clockwise instead of counterclockwise, and by suitably alternating our choices we can keep the degree of spiralling bounded. $\square$

Proof of theorem 4.0.3 Let $z_\infty = \lim z_n$ where the z_n are the series of cusp points in the previous proof (in fact, we will add an additional requirement on z_n later in this proof). Writing $L_n(z)$ as before, and $L_\infty(z) = \log(z - z_\infty)$ defined in the same way, we get $L_\infty(z) - L_n(z) = \log(1 + w_n(z))$ where $w_n(z) = (z_n - z_\infty)/(z - z_n)$. If we can find a sequence ζ_n such that $\zeta_n \rightarrow z_\infty$, $\text{Im } L_n(\zeta_n)$ is unbounded and $\log(1 + w_n(\zeta_n))$ is bounded then $\text{Im } L_\infty(\zeta_n)$ will be unbounded, which would prove that z_∞ was a point of infinite spiralling on the boundary. To show that $\log(1 + w_n(\zeta_n))$ is bounded it is enough to find a sequence such that $|w_n(\zeta_n)| < 1/2$, or equivalently that $|\zeta_n - z_n| > 2|z_n - z_\infty|$. For $|z - z_n| \leq \eta$, for some $\eta > 0$ depending on n , we know that $\text{Im } L_n(z) > a + b\eta$ for some constants $a, b > 0$. (This just follows from the previous proof.) Now, if $|z_n - z_\infty| < \eta/2$ then we can choose a point ζ_n a distance η from z_n , and this sequence satisfies $|w_n(\zeta_n)| < 1/2$. The additional requirement on z_n mentioned at the start of this proof is that $|z_n - z_\infty| < \eta/2$ which can be ensured by always choosing the next neighbouring cusp in the sequence (z_n) sufficiently close to the previous one.

The set of points where the spiralling is infinite is clearly dense (this procedure can be started as close to any point as you like). Moreover, the number of limit points you can get to is uncountable. For any given choice of the first m points in the sequence (z_n) , there is a countably infinite number of choices for the next point z_{m+1} (this is the comment in the proof of theorem 4.0.2 that says there are a countably infinite number of choices for what was called z_2 in that proof). Here we will assume that the point z_1 is common to all such sequences. Even with this restriction we get an uncountable number of limit points. Each choice of sequence (z_n) gives rise to a unique limit z_∞ . This is because the continued fraction expansion of an irrational is unique. Given a sequence (z_n) define the sequence (r_n) to be the rational numbers associated to the cusps (so that $z_n = \mu(r_n)$). The sequence (r_n) will be the sequence of continued fraction partial approximates to $\beta = \lim r_n = \mu^{-1}(z_\infty)$ (because z_{n+1} is a neighbour of z_n). Since each irrational β has a unique continued fraction approximation, different sequences will give rise to different limits. Let S be the set of limits z_∞ coming about in this way, we have shown that there is a bijection between S and the set of countably infinite sequences of natural numbers $\mathbb{N}^{\mathbb{N}}$. Explicitly, this bijection is as follows. Given a finite sequence $(z_n)_{n=1}^m$, there is a countably infinite set $Z((z_n)_{n=1}^m)$ of choices for the next point z_{m+1} . For each such sequence, choose a bijection $\pi((z_n)_{n=1}^m) : Z((z_n)_{n=1}^m) \rightarrow \mathbb{N}$. Given a sequence $(a_n) \in \mathbb{N}^{\mathbb{N}}$ we define a sequence z_n inductively as follows. The first point z_1 is always the same. Given the first m points, $(z_n)_{n=1}^m$ we define $z_{m+1} = \pi((z_n)_{n=1}^m)^{-1}(a_{m+1})$. The bijection between $\mathbb{N}^{\mathbb{N}}$ and S sends a sequence $(a_n) \in \mathbb{N}^{\mathbb{N}}$ to the limit z_∞ of this sequence. This is surjective by definition, and injective because such sequences give rise to unique limits. Since $\mathbb{N}^{\mathbb{N}}$ is uncountable, so is S . The character

of the set of points about which this result proves there is infinite spiralling, is somewhat akin to a countable union of Cantor sets. $\square$

Chapter 5

Conjectural picture of the boundary

We have proved that the boundaries of the Maskit and Bers slices spiral infinitely at an uncountable, dense set. However, there is very good numerical evidence to suggest that in fact they spiral infinitely at almost every point in their boundaries. We present an argument for this based on two unproven conjectures (sections 5.3 and 5.4). Our argument relies on some facts from complex analysis (section 5.1) and number theory (section 5.2).

Throughout this chapter and the next, we refer to `C++` and `Mathematica` files and functions. These are included on CD attached to this thesis. They are also included on the author's web page, which at the time of writing is maths.thesamovar.net.

5.1 Twisting

Let $\mathbb{D} \subseteq \mathbb{C}$ be the unit disc in $\mathbb{C}$ and $f : \mathbb{D} \rightarrow G \subseteq \mathbb{C}$ conformal. We define two sorts of behaviour at the boundary.

A *Stolz angle* at $\zeta \in \partial\mathbb{D}$ is any set of points $\Delta \subseteq \mathbb{D}$ a bounded hyperbolic distance from the radius $[0, \zeta]$. We say f has the *angular limit* $a \in \hat{\mathbb{C}}$ at $\zeta \in \partial\mathbb{D}$ if

$$\lim_{z \rightarrow \zeta, z \in \Delta} f(z) = a \quad (5.1)$$

for every Stolz angle Δ at ζ . We will write $f(\zeta)$ for the angular limit a . We say f is *conformal* at $\zeta \in \partial\mathbb{D}$ if

$$f'(\zeta) = \lim_{z \rightarrow \zeta, z \in \Delta} \frac{f(z) - f(\zeta)}{z - \zeta} = \lim_{z \rightarrow \zeta, z \in \Delta} f'(z) \neq 0, \infty \quad (5.2)$$

for every Stolz angle Δ at ζ . We say that f is *twisting* at ζ if the angular limit $f(\zeta) \neq \infty$ exists and

$$\liminf_{z \rightarrow \zeta, z \in \Gamma} \arg(f(z) - f(\zeta)) = -\infty, \quad \limsup_{z \rightarrow \zeta, z \in \Gamma} \arg(f(z) - f(\zeta)) = +\infty$$

for every curve $\Gamma \subseteq \mathbb{D}$ ending at ζ . We say that $f(\zeta)$ is *sectorially accessible* from G if G contains an open triangle with vertex $f(\zeta)$. We define $\text{Sect}(f)$ to be the set of all $\zeta \in \partial\mathbb{D}$ such that $f(\zeta)$ is sectorially accessible. Clearly if $f(\zeta)$ is sectorially accessible it cannot be twisting. We write λ for linear measure (one dimensional Lebesgue measure) on subsets of $\mathbb{C}$.

Essentially (that is, up to sets of measure 0) we think of sectorially accessible and conformal as the same, and the opposite of twisting. This is the intuitive content of theorem 5.1.1 below. See [McMillan69] and chapter 6 of [Pomm92].

Theorem 5.1.1 (McMillan Twist Theorem) *At almost all $\zeta \in \partial\mathbb{D}$ the map f is either conformal or twisting.*

The twist theorem implies that $\text{Sect}(f)$ differs from the set of conformal boundary points by a set of measure 0.

5.2 Continued fractions

We need some results about continued fractions. These follow quite simply by applying the ergodic theorem to the Gauss measure and the continued fraction transformation.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a *probability space* and $T : \Omega \rightarrow \Omega$ a *measure preserving transformation*. That is, $\mathcal{F}$ is a σ -algebra of subsets of Ω , $\mathbb{P}$ is a probability measure on $\mathcal{F}$, and T is a measurable function satisfying $\mathbb{P}(T^{-1}A) = \mathbb{P}(A)$ for all $A \in \mathcal{F}$. A set $A \in \mathcal{F}$ is *invariant* if $T^{-1}A = A$. A measurable function g on Ω is said to be *invariant* if $g(T\omega) = g(\omega)$ a.e. The transformation T is *ergodic* if every invariant set has measure 0 or 1. If $f : \Omega \rightarrow \mathbb{R}$ is integrable we define the *expectation* of f to be $\mathbb{E}[f] = \int f \, d\mathbb{P}$.

Theorem 5.2.1 (The Ergodic Theorem) *If f is integrable, then there exists an integrable, invariant function $\hat{f}$ such that $\mathbb{E}[\hat{f}] = \mathbb{E}[f]$ and*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k \omega) = \hat{f}(\omega) \text{ a.e.}$$

If T is ergodic then $\hat{f}(\omega) = \mathbb{E}[f]$ a.e.

For a proof of the ergodic theorem, see [Bill65]. As a simple application we get the following.

Corollary 5.2.2 *Let T be ergodic and $f = I_A$ the indicator function of A . Then*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} I_A(T^k \omega) = \mathbb{P}(A) \text{ a.e.}$$

We apply this to prove several results about continued fractions. We consider the unit interval $[0, 1]$ with the *Gauss measure* defined by

$$\mathbb{P}(A) = \frac{1}{\log 2} \int_A \frac{dx}{1+x}.$$

This measure is absolutely continuous with respect to Lebesgue measure so the terms integrable and a.e. apply equally to the Gauss measure or Lebesgue measure. We define the function

$$T(\omega) = \begin{cases} \{1/\omega\} & \text{if } \omega \neq 0, \\ 0 & \text{if } \omega = 0. \end{cases}$$

Here $\{x\}$ means the fractional part of x , and $[x]$ means the integer part. This function T is ergodic with respect to the Gauss measure, see [Bill65]. We also define the *partial quotients*

$$a(\omega) = \begin{cases} [1/\omega] & \text{if } \omega \neq 0, \\ \infty & \text{if } \omega = 0 \end{cases}$$

and

$$a_n(\omega) = a(T^{n-1}\omega).$$

We use the notation $[a_1 \ a_2 \ a_3 \ \dots]$ to mean the continued fraction

$$\frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}.$$

We define $p_n(\omega)$ and $q_n(\omega)$ to be the numerator and denominator of $[a_1 \dots a_n]$. We get the following

relations for $n \geq 1$,

$$\begin{aligned} p_{n+1} &= a_{n+1}p_n + p_{n-1} \\ q_{n+1} &= a_{n+1}q_n + q_{n-1} \end{aligned} \tag{5.3}$$

If we specify $p_{-1}/q_{-1} = 1/0$ and $p_0/q_0 = 0/1$ these relations can be used to define p_n and q_n . From this, it is straightforward to prove (see [Bill65] chapter 1, section 4) that

$$\frac{1}{q_n(q_n + q_{n+1})} \leq \left| \omega - \frac{p_n}{q_n} \right| \leq \frac{1}{q_n q_{n+1}}.$$

From this, it is easy to prove the following result, which we will use later,

$$\frac{1}{(a_{n+2} + 1)(a_{n+1} + 1) + 1} \leq \left| \omega - \frac{p_{n+1}}{q_{n+1}} \right| \Bigg/ \left| \omega - \frac{p_n}{q_n} \right| \leq \frac{a_{n+1} + 2}{a_{n+1}^2 a_{n+2}}. \tag{5.4}$$

Let f be the indicator of the set $\{\omega : a_1(\omega) = k\}$. Applying corollary 5.2.2 we see that the asymptotic relative frequency of k among the partial quotients is

$$\frac{1}{\log 2} \int_{1/(k+1)}^{1/k} \frac{dx}{1+x} = \frac{1}{\log 2} \log \frac{(k+1)^2}{k(k+2)}.$$

In particular, since the right hand side is nonzero for all k , k appears infinitely often in the continued fraction expansion of almost all ω . This also shows that the partial quotients are unbounded for almost all ω .

By a similar method, we can find the asymptotic relative frequency of the sequence $k_1, k_2, \dots, k_m$. We let f be the indicator of the set

$$A(k_1, \dots, k_m) = \{\omega : a_1(\omega) = k_1, \dots, a_m(\omega) = k_m\},$$

and apply corollary 5.2.2 to get the asymptotic relative frequency of the sequence to be

$$\mathbb{P}(A(k_1, \dots, k_m)) = \frac{1}{\log 2} \int_{A(k_1, \dots, k_m)} \frac{dx}{1+x}.$$

In fact, all we need here is that this probability and asymptotic relative frequency is nonzero.

Lemma 5.2.3 *Every sequence $(k_1, \dots, k_m)$ occurs infinitely often in the continued fraction expansion of almost all ω .*

The following results are also proved in [Bill65] by relatively simple applications of the ergodic theorem.

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log q_n(\omega) = \frac{\pi^2}{12 \log 2} \quad \text{a.e.}$$

Roughly, we think of this as saying that for large n and almost all ω ,

$$q_n(\omega) \sim \beta^n,$$

where $\beta = e^{\frac{\pi^2}{12 \log 2}} \approx 3.28$. It also follows that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \left| \omega - \frac{p_n(\omega)}{q_n(\omega)} \right| = -\frac{\pi^2}{6 \log 2} \quad \text{a.e.}$$

Again, we think of this as saying

$$\left| \omega - \frac{p_n(\omega)}{q_n(\omega)} \right| \sim \gamma^n,$$

where $\gamma = e^{-\frac{\pi^2}{6 \log 2}} \approx 0.09$.

We also have results on Diophantine approximation. It is possible to prove the following directly.

Lemma 5.2.4 *For all irrational $\omega \in [0, 1]$, we have*

$$\left| \omega - \frac{p_n}{q_n} \right| < \frac{1}{q_n^2}.$$

We do not use the rest of these results directly, but they are important results in the theory of continued fractions and Diophantine approximation which help to give a better intuitive picture of what is going on.

Theorem 5.2.5 *The event $a_n(\omega) > \alpha_n$ occurs infinitely often with probability 0 if $\sum 1/\alpha_n$ converges and probability 1 if it diverges.*

Theorem 5.2.6 *Let $f(q) : \mathbb{N} \rightarrow \mathbb{R}^+$.*

1. *If $qf(q)$ is nonincreasing and $\sum f(q) = \infty$ then for almost all ω we have*

$$\left| \omega - \frac{p}{q} \right| < \frac{f(q)}{q}$$

for infinitely many p and q .

2. *If $\sum f(q) < \infty$ then for almost all ω the inequality holds for only finitely many p and q .*

In particular, setting $f(q) = \frac{1}{q \log q \log \log q}$ and $f(q) = \frac{1}{q(\log q)^2}$, whose sums diverge and converge respectively, we have the following.

Corollary 5.2.7 *For almost all ω , for infinitely many p and q we have*

$$\frac{1}{q^2(\log q)^2} < \left| \omega - \frac{p}{q} \right| < \frac{1}{q^2 \log q \log \log q}.$$

To show these sums converge or diverge we apply Cauchy's integral test which states in this case that $\sum_{q=1}^{\infty} f(q)$ converges or diverges if $\int_1^{\infty} f(q) dq$ converges or diverges. The sum of $1/q \log q$ diverges because the integral of $1/q \log q$ is $\log \log q$ which tends to infinity as $q \rightarrow \infty$. Substituting $u = \log q$ we see that the integral of $1/q \log q \log \log q$ is the integral of $1/u \log u$ and so also diverges. Similarly, the sum of $1/q(\log q)^2$ converges because the integral is $-1/\log q$ which tends to 0 as $q \rightarrow \infty$.

For more details on continued fractions, see [Bill65] and [HarWri79].

5.3 Uniform local coordinates

Our argument in section 5.5.2 requires a somewhat technical result. Consider the following situation. Take the p/q -cusp in the Maskit slice, and let μ be the usual coordinates, and $t = \text{Tr}_{p/q}(\mu)$ be the trace coordinates in a small neighbourhood. You can see in figure 5.1 that the linear part of $\text{Tr}_{p/q}(\mu)$ is quite a good estimate of $\text{Tr}_{p/q}(\mu)$ in a small region surrounding the cusp $\mu_{p/q}$. We have good numerical evidence to support the following conjecture which is an attempt to capture this.

Conjecture 5.3.1 (Uniform Local Coordinates) *There exists $\epsilon > 0$ such that for all p/q , for the region $|t - 2| \leq \epsilon$, we have*

$$\frac{1}{2} \frac{|t - 2|}{|\text{Tr}'_{p/q}(\mu_{p/q})|} \leq |\mu - \mu_{p/q}| \leq \frac{3}{2} \frac{|t - 2|}{|\text{Tr}'_{p/q}(\mu_{p/q})|}.$$

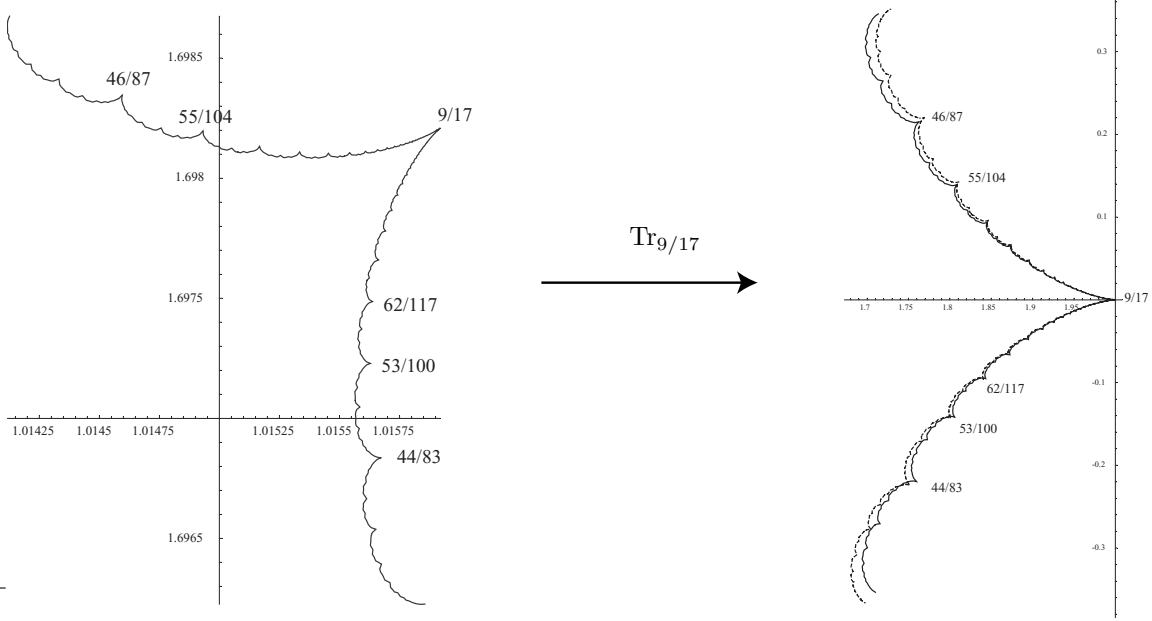


Figure 5.1: Trace coordinates. The figure on the left hand side is a small region in the Maskit slice surrounding the $9/17$ -cusp. The solid line on the figure on the right hand side is the same region in the $\text{Tr}_{9/17}$ trace coordinates. The dashed line is the same region if you only apply the linear part of the function $\text{Tr}_{9/17}$. The fractions on both figures correspond to the cusps.

The purpose of this conjecture will become clear in section 5.5.2. However, roughly speaking it says that if you can estimate $|\text{Tr}'_{p/q}(\mu_{p/q})|$ and $\text{Tr}_{p/q}(\mu)$ then you can estimate $|\mu - \mu_{p/q}|$. Not having this conjecture or something similar is a surprisingly large barrier to proving results about the Maskit slice. For example, lemma 4.2.1 tells us that the sequence of neighbouring cusps $\mu_n \rightarrow \mu_\infty = \mu_{p/q}$ is an approximate cusp sequence, that is it satisfies

$$\mu_n = \mu_\infty - \frac{\pi^2}{n^2 \text{Tr}'_\infty(\mu_\infty)} + O(n^{-3}).$$

The problem is that the implicit constant in the $O(n^{-3})$ term depends on p/q in a complicated way. However, if we write $t_n = \text{Tr}_{p/q}(\mu_n)$ then we know that

$$t_n = 2 - \frac{\pi^2}{n^2} + O(n^{-3}),$$

where the $O(n^{-3})$ term depends on universal constants independent of p/q . If the conjecture is correct then we can say that

$$|\mu_n - \mu_\infty| = \Theta\left(\frac{\pi^2}{n^2 |\text{Tr}'_\infty(\mu_\infty)|}\right),$$

where the implicit constants are universal (don't depend on p/q). Note that the $\Theta(-)$ notation here is similar to the $O(-)$ notation, and is defined in appendix A. Roughly speaking $f = O(g)$ means that f is of the order of g or smaller, whilst $f = \Theta(g)$ means that f is of the same order as g .

The numerical evidence for this conjecture is as follows. Assume for the moment that $\text{Tr}_{p/q}$ is invertible in the region we are considering. Let

$$E_{p/q}(t) = \mu - \mu_{p/q} - \frac{t - 2}{\text{Tr}'_{p/q}(\mu_{p/q})}$$

be the error term (consisting of the second and higher order terms in the series expansion of $\text{Tr}_{p/q}^{-1}$). We want to show that

$$|E_{p/q}(t)| \leq \frac{|t-2|}{2|\text{Tr}'_{p/q}(\mu_{p/q})|}.$$

So write

$$\frac{\text{Tr}'_{p/q}(\mu_{p/q})E_{p/q}(t)}{t-2} = \sum_{n=2}^{\infty} a_n(t-2)^{n-1}.$$

Now if $|t-2| \leq 1$ then this sum will be less than $\sum_{n=2}^{\infty} |a_n|$. So if this sum is always less than $1/2$ the conjecture follows. Numerical evidence suggests that this is the case. This evidence only includes cases up to $q \leq 20$ because of numerical stability problems with Mathematica.

An alternative way of showing numerically that $E_{p/q}(t)$ is bounded in this way is to use Cauchy's theorem

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta-z)^{k+1}} d\zeta.$$

Suppose C is a circle about z of radius ϵ and $|f(z)| \leq A$ on C , then

$$\left| \frac{f^{(k)}(z)}{k!} \right| \leq \frac{A}{\epsilon^k}.$$

Now suppose that f in the above is defined by $f(z) = \text{Tr}_{p/q}^{-1}(t) - \mu_{p/q}$ and $z = t-2$. If we could find a constant B not depending on p/q such that on C we have $|f(z)| \leq B/|\text{Tr}'_{p/q}(\mu_{p/q})|$ then the conjecture follows. For $k \geq 2$, the quantity $f^{(k)}(0)/k!$ is the k th coefficient of the Taylor series of $E_{p/q}(t)$. It follows that when $|t-2| \leq \eta\epsilon$

$$\left| \frac{\text{Tr}'_{p/q}(\mu_{p/q})E_{p/q}(t)}{t-2} \right| \leq \sum_{k=2}^{\infty} \frac{A|\text{Tr}'_{p/q}(\mu_{p/q})|\eta^{k-1}\epsilon^{k-1}}{\epsilon^k} = \frac{B\eta}{\epsilon(1-\eta)}.$$

By choosing η sufficiently small we can ensure that this is always less than $1/2$.

The numerical evidence for the existence of this universal bound is very good indeed. It is difficult to compute $\text{Tr}_{p/q}^{-1}$ so we take the following approach (see figure 5.2). For every p/q let the circle D be the circle of radius $1/|\text{Tr}'_{p/q}(\mu_{p/q})|$ around $\mu_{p/q}$. It turns out that for all $q \leq 1000$ the image under $\text{Tr}_{p/q}$ of D is always contained in an annulus which seems to have universal inner and outer radii (about 0.87 and 1.15 respectively). This analysis is carried out by the C++ function `view::imagecircleanalysis` in the file `maskitalgorithms.cpp`. Since $f'(z) = (\text{Tr}_{p/q}^{-1})'(t) = 1/\text{Tr}'_{p/q}(\mu)$, $f'(z) \neq 0$ and so f cannot have a local maximum within C , the circle of radius 0.87 about $t=2$. This implies the existence of a universal bound of the required type (by the maximum modulus principle).

Finally, we note that numerical evidence suggests that on the disc E of radius $1/|\text{Tr}'_{p/q}(\mu_{p/q})|$ about $\mu_{p/q}$, the function $\text{Tr}_{p/q}$ is invertible. Firstly, a small number of sample calculations suggest that the image of E is always simply connected. We provide no numerical evidence in support of this, other than to say that in a small number of test cases it seemed to be the case, and it is very implausible that it is not the case. However, more extensive computations (for all $q \leq 80$) suggest strongly that on E , $\text{Tr}'_{p/q}(\mu) \neq 0$. By analytic continuation, this would imply that $\text{Tr}_{p/q}$ is invertible on E . See the Mathematica notebook `invertibility-regions.nb` for these calculations.

In conclusion, the evidence for this conjecture is far from complete (in particular, because the argument above relies on the invertibility of $\text{Tr}_{p/q}$ in the required region, for which the evidence is relatively weak), but it is nonetheless strong enough to make it likely to be true.

Probably the best method to try to prove this conjecture would be to first prove some statement about the trace functions for all p/q using induction on the level of p/q and the formula $\text{Tr} W_{(p+r)/(q+s)} = \text{Tr} W_{p/q} \cdot \text{Tr} W_{r/s} - \text{Tr} W_{(r-p)/(s-q)}$ (see section 6.2.1). The level of p/q is the number of steps needed to get to p/q using Farey addition (see section 6.2.1). The second step would be to derive the conjecture from this statement about the trace functions for all p/q . The proof would probably be relatively straightforward, the difficult thing is working out precisely what statement to prove using induction.

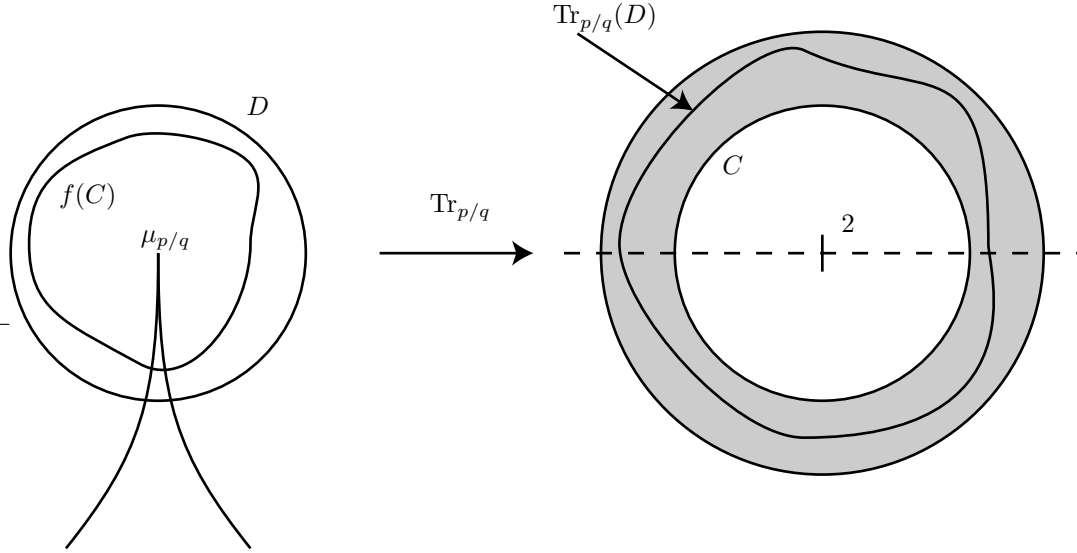


Figure 5.2: Showing numerically that $\text{Tr}_{p/q}$ is approximately linear

5.4 Trace derivatives

The argument in section 5.5.2 also requires one more result, conjecture 5.4.3 below. However, this section also allows us to make a conjecture about the Hausdorff dimension of the Maskit slice in section 5.6.

Define $T_{p/q} = |\text{Tr}'_{p/q}(\mu_{p/q})|$. Given $\omega \in [0, 1]$ and $p_n/q_n \rightarrow \omega$ the partial approximants, define $T_n = T_n(\omega) = T_{p_n/q_n}$. We make several conjectures about the statistical and limiting properties of $T_{p/q}$ on the basis of numerical evidence. See the Mathematica notebook `trace-derivatives.nb`.

We first argue that for almost all ω , for large n , $\log T_n / \log q_n \in [1.69846, 2.35407]$. First, we note that there is strong statistical evidence to suggest that for all ω and for all n , $T_n/T_{n-1} \in [T_{1/a_n}, T_{1/a_{n+1}}]$. The following table gives the minimum and maximum values of T_n/T_{n-1} for 30000 samples subject to the restriction that $q_n \leq 2000$.

$a_n(\omega)$	$\min T_n/T_{n-1}$	$\max T_n/T_{n-1}$	T_{1/a_n}	$T_{1/a_{n+1}}$	Samples
1	1.	3.4641	1.	3.4641	30718
2	3.4641	6.97359	3.4641	6.97359	12174
3	6.97359	11.875	6.97359	11.875	6666
4	11.875	18.9384	11.875	18.9384	4084
5	18.9384	28.9523	18.9384	28.9523	2872
6	28.9523	42.6073	28.9523	42.6073	2098
7	42.6073	60.5356	42.6073	60.5356	1605
8	60.5356	83.3487	60.5356	83.3487	1239
9	83.3487	111.652	83.3487	111.652	1025
10	111.652	146.05	111.652	146.05	778
11	146.05	187.148	146.05	187.148	713
12	187.148	235.55	187.148	235.55	549
13	235.55	291.864	235.55	291.864	508
14	291.864	356.695	291.864	356.695	459
15	356.695	430.651	356.695	430.651	382
16	430.651	514.337	430.651	514.337	294
17	514.337	608.363	514.337	608.363	320
18	608.363	713.334	608.363	713.334	288
19	713.334	829.858	713.334	829.858	232
20	829.858	958.543	829.858	958.543	225

As noted, this suggests the following conjecture.

Conjecture 5.4.1 *For all ω, n , we have that $|\text{Tr}'_n|/|\text{Tr}'_{n-1}| \in [\text{Tr}'_{1/a_n}, \text{Tr}'_{1/a_{n+1}}]$.*

Now, the event $a_n(\omega) = m$ occurs with relative frequency

$$p_m = \frac{1}{\log 2} \log \frac{(m+1)^2}{m(m+2)}.$$

Consider the first n partial quotients of ω . In the limit, np_m of these will be m . Therefore, very roughly speaking, for almost all ω , in the limit we will have

$$\prod_{m=1}^{\infty} T_{1/m}^{np_m} \leq T_n \leq \prod_{m=1}^{\infty} T_{1/(m+1)}^{np_m}.$$

Taking logs and dividing by n we get

$$\sum_{m=1}^{\infty} p_m \log T_{1/m} \leq \log T_n/n \leq \sum_{m=1}^{\infty} p_m \log T_{1/(m+1)}.$$

Evaluating these sums numerically using the values of $T_{1/m}$ from $m = 1$ to $m = 2000$ gives $\log T_n/n \in [2.01533, 2.79327]$. We already know that $q_n \sim \beta^n$ where $\log \beta = \pi^2/12 \log 2$. Combining these gives the limiting value of $\log T_n/\log q_n \in [1.69846, 2.35407]$ for almost all ω .

Evaluating $\log T_{p/q}/\log q$ for all p/q with $q \leq 2000$ gives the histogram in figure 5.3. As you can see, there are many values outside the limiting range $[1.69846, 2.35407]$, but that the majority are within this range, which is what we would expect. The probability p_m is only a limiting relative frequency, therefore it is possible that for any particular n , $\log T_n/\log q_n$ might be outside these bounds. The constant term introduced into the size of T_n will vanish in the limit when we take logs and divide by $\log q_n$ (which tends to infinity).

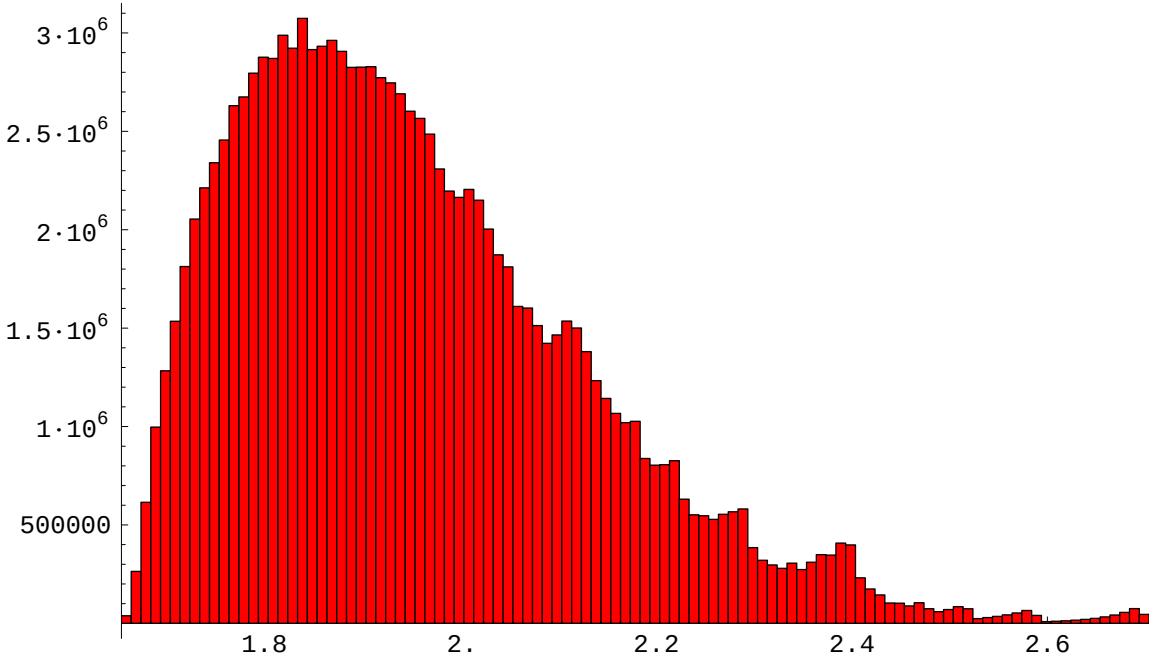


Figure 5.3: Histogram of values of $\log T_{p/q}/\log q$ for $q \leq 2000$

Conjecture 5.4.2 *Let $p_n/q_n \rightarrow \omega \in [0, 1]$ be the sequence of partial approximants. Let $T_n = T_{p_n/q_n}$.*

Then for almost all ω , we have

$$\liminf_n \frac{\log T_n}{\log q_n} \geq 1.69$$

and

$$\limsup_n \frac{\log T_n}{\log q_n} \leq 2.36.$$

It may even be possible that something considerably stronger than this is true. If we choose $\omega \in [0, 1]$ uniformly at random and plot $\log T_n$ against $\log q_n$ we see that all of these graphs are very close to being straight lines. Figure 5.4 shows these graphs for nine randomly selected ω up to $q \leq 2000$. The shaded area is the region $\log T_n / \log q_n \in [1.69846, 2.35407]$, and the black line is the best fit. The first conjecture states that eventually all the points will be contained in this shaded area, the stronger conjecture says that eventually the points will tend toward being a straight line contained in this area. This stronger conjecture is equivalent to saying that

$$\liminf_n \frac{\log T_n}{\log q_n} = \limsup_n \frac{\log T_n}{\log q_n}.$$

Equivalently, $T_n \sim q_n^\alpha$ for some $\alpha \in [1.69846, 2.35407]$.

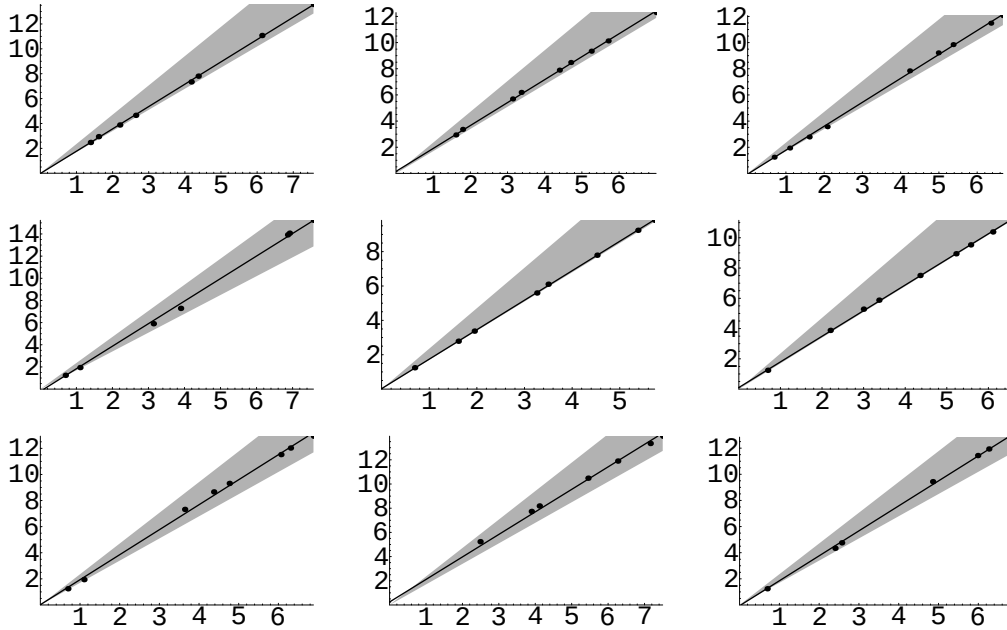


Figure 5.4: Nine graphs of $\log T_n$ against $\log q_n$

Running a linear regression on 10000 randomly chosen ω gives us a correlation coefficient of $r^2 \geq 0.985$ with an average r^2 of 0.998. The estimated variance was at most 0.47 and the average estimated variance was 0.045. These numbers are very good. Unfortunately, these linear regressions could only be applied to graphs with at most 15 points because of the requirement that $q \leq 2000$. This evidence is considerably less convincing than the previous evidence.

More likely is that almost all the time $\log T_n / \log q_n$ varies in the interval $[1.69846, 2.35407]$, mostly tending towards the bottom end when the common low integers appear in the continued fraction expansion, with occasional large jumps towards the top end when the infrequent large integers appear.

We could define the *exponent distribution interval* E_ω of $\omega \in [0, 1]$ to be the interval

$$E_\omega = [\liminf_n \frac{\log T_n}{\log q_n}, \limsup_n \frac{\log T_n}{\log q_n}].$$

The first conjecture states that for almost all ω , $E_\omega \subseteq [1.69846, 2.35407]$, the second states that

for almost all ω , $|E_\omega| = 1$. Numerical evidence suggests strongly that for all ω , $E_\omega \subseteq [1.6, 3]$. Even stronger than this last statement, the numerical evidence suggests the following conjecture.

Conjecture 5.4.3 *For all p/q , $q^{1.6} \leq |\text{Tr}'_{p/q}| \leq q^3$.*

5.5 Spiralling almost everywhere

We propose the following conjecture.

Conjecture 5.5.1 *The boundaries of the Maskit and Bers slices spiral infinitely almost everywhere.*

There are two ways of seeing why this is probably true. The first is more intuitively suggestive, but less workable. The second is less intuitive but only requires the very plausible conjectures 5.3.1 and 5.4.3 to make it work.

5.5.1 Random walk

Define the function

$$\theta : [0, 1] \times \mathbb{N} \longrightarrow \mathbb{R}; (\omega, n) \mapsto \text{sp. deg } \wp_{p_n(\omega)/q_n(\omega)}$$

sending (ω, n) to the spiralling degree (definition 4.1.1) of the pleating ray of the n th partial approximant of ω . Essentially, the proof of theorem 4.0.2 shows that for certain ω , $\theta(\omega, n) \rightarrow \infty$ as $n \rightarrow \infty$. The proof of theorem 4.0.3 shows that for these ω , if $\theta(\omega, n) \rightarrow \infty$ then the boundary is spiralling infinitely at $f(\omega)$. Although we have not proved it, it seems intuitively plausible that for any ω , if $\sup_n \theta(\omega, n) = +\infty$ and $\inf_n \theta(\omega, n) = -\infty$ then the boundary is twisting at $f(\omega)$.

Define the random variable Θ_n on the state space $[0, 1]$ with either Lebesgue measure or Gauss measure by $\Theta_n(\omega) = \theta(\omega, n)$. If it were the case that $\Delta\Theta_n := \Theta_{n+1} - \Theta_n$ were independent, identically distributed random variables (distributed like $\Delta\Theta$ say), then it would almost immediately follow that the boundary is twisting almost everywhere. As long as $\mathbb{P}(\Delta\Theta > \epsilon) > 0$ and $\mathbb{P}(\Delta\Theta < -\epsilon) > 0$ for some ϵ , then the Θ_n would be a nontrivial one dimensional random walk and would therefore get close to $\pm\infty$ infinitely often.

Unfortunately, there are two problems. Firstly, the $\Delta\Theta_n$ are not independent. Secondly, we would need to prove that $\sup_n \theta(\omega, n) = +\infty$ and $\inf_n \theta(\omega, n) = -\infty$ together imply that the boundary is twisting at $f(\omega)$. The second problem is probably not too difficult, but the first is. Even though the $\Delta\Theta_n$ are not independent, it seems on the basis of numerical work that they are sufficiently close to independent and identically distributed that the conclusion should be true.

If we could show that there was a uniform N such that whenever the n th partial approximant satisfied $a_n(\omega) > N$ we had $|\Delta\Theta_n(\omega)| > \epsilon$, then this would be enough. Unfortunately, the trace derivative formula, equation 4.13, involves terms that depend crucially on $p_n(\omega)/q_n(\omega)$, and so finding such a uniform N cannot be guaranteed. Again, numerically it seems as though quite small choices of N do in fact suffice ($N = 3$ or $N = 4$ seems to work).

The numerical basis for these claims is as follows. The program (Mathematica notebook `random-walk.nb`) was instructed to randomly choose 30000 different $\omega \in [0, 1]$ (with a uniform distribution). For each ω , the program runs through the continued fraction expansion as far as numerical precision allows and with the restriction that $q_n(\omega) < 2000$. Whenever it encounters $a_n(\omega) = N$ it stores the value of $|\Delta\Theta_n(\omega)|$. This process was repeated for various values of N between 1 and 20, and the minimum and maximum values for $|\Delta\Theta_n(\omega)|$ are summarised in the second and third columns of the table below. The fifth and sixth columns summarise the same data for the case $a_n(\omega) \geq N$. Note that the maximum values in the sixth column are close to the maximum possible value of $\pi/2 \approx 1.57$. Also note that the difference between the second and third columns gets quite small as N increases, which gives some plausibility to the idea that the $\Delta\Theta_n$ are approximately identically distributed for large $a_n(\omega)$.

N	$\min_{=}\Delta\Theta_n(\omega)$	$\max_{=}\Delta\Theta_n(\omega)$	Samples	$\min_{\geq}\Delta\Theta_n(\omega)$	$\max_{\geq}\Delta\Theta_n(\omega)$	Samples
1	0	0.229895	30800	0	1.56769	71787
2	7.4455×10^{-10}	0.151893	12125	7.4455×10^{-10}	1.56769	40987
3	0.0519555	0.335039	6633	0.0519555	1.56769	28862
4	0.259119	0.512935	4081	0.259119	1.56769	22229
5	0.449701	0.662867	2793	0.449701	1.56769	18148
6	0.606954	0.782329	2058	0.606954	1.56769	15355
7	0.731879	0.877532	1609	0.731879	1.56769	13297
8	0.830784	0.953197	1234	0.830784	1.56769	11688
9	0.910019	1.01443	972	0.910019	1.56769	10454
10	0.974477	1.0654	828	0.974477	1.56769	9482
11	1.02777	1.10792	741	1.02777	1.56769	8654
12	1.07248	1.14399	588	1.07248	1.56769	7913
13	1.11046	1.17471	484	1.11046	1.56769	7325
14	1.1431	1.2014	459	1.1431	1.56769	6841
15	1.17145	1.22451	404	1.17145	1.56769	6382
16	1.19631	1.24532	361	1.19631	1.56769	5978
17	1.21828	1.2635	308	1.21828	1.56769	5617
18	1.23778	1.27983	282	1.23778	1.56769	5309
19	1.25527	1.29463	254	1.25527	1.56769	5027
20	1.27101	1.30804	215	1.27101	1.56769	4773

There are various numerical issues which need to be considered here. First of all, it may be that the number of samples is insufficient to estimate the parameters. We do not rigorously consider this possibility, but repeating the process with 1000 samples or with 8000 rather than 30000 samples gives the same numbers to almost two significant figures which suggests that they are reasonably accurate. The other possibility is that these numbers are an artifact of the small denominators $q_n(\omega) < 2000$ being considered. This problem is particularly acute for large N . Consider that when we encounter $a_n(\omega) = N$ we get that $q_{n+1}(\omega) \geq Nq_n(\omega)$. This is partially reflected in the table by the fact that we have considerably fewer samples for large N than for small N . However, repeating the experiment with the conditions that $0 < q_n(\omega) < 200$, $200 < q_n(\omega) < 700$ and $700 < q_n(\omega) < 2000$ gives the same values to almost two significant figures. Again, this lends some support to these numbers being universal.

In summary, this approach seems to lend quite strong numerical support to conjecture 5.5.1. However, it does not suggest any obvious ways to go about proving it.

5.5.2 Boundary conformality

Let $f : \mathbb{D} \rightarrow \mathcal{M}$ be the Riemann map from the unit disc $\mathbb{D}$ to the Maskit slice $\mathcal{M}$ (or the Bers slice). If we can show directly that for almost all points $\zeta \in \partial\mathbb{D}$ that f is not conformal (that is, equation 5.2 doesn't hold), then by the twist theorem (theorem 5.1.1), the map f must be twisting at almost all $\zeta \in \partial\mathbb{D}$. That f is twisting is equivalent to what we earlier called the boundary spiralling infinitely almost everywhere.

Equation 5.2 implies that if f is conformal at $\zeta \in \partial\mathbb{D}$, $z_n \in \Delta$ and $z_n \rightarrow \zeta$ then

$$\lim_{n \rightarrow \infty} \frac{f(z_n) - f(\zeta)}{z_n - \zeta} \quad (5.5)$$

must exist and the terms of the sequence must be bounded away from 0 and ∞ . We suggest that for almost all ζ there is a sequence z_n for which the terms of the sequence are not bounded away from 0 and ∞ . We give the details of this sequence below, and assuming conjectures 5.3.1 and 5.4.3 prove that it has the required properties.

Given $\omega \in [0, 1]$, define a_n and p_n/q_n to be the sequence of partial quotients and partial approximants of ω . Now define

$$\alpha_n = \left| \omega - \frac{p_n}{q_n} \right|,$$

$$z_n = p_n/q_n + i\epsilon|\omega - p_n/q_n|$$

for some small ϵ , and

$$\sigma_n = \left| \frac{f(\omega) - f(z_n)}{\omega - z_n} \right|.$$

Note that the z_n are in a Stolz angle at ω of angle $\tan^{-1} \epsilon$. also note that $|\omega - z_n| \approx \alpha_n$. We show, assuming conjecture 5.3.1, that for almost all ω there is a subsequence n_m with the property that $\sigma_{n_m+m}/\sigma_{n_m} \leq 1/2^m$. Clearly, this contradicts the sequence (σ_n) being bounded away from 0 and ∞ .

Fix N (we specify how exactly later). Write S_N^{m+1} for the sequence $(N \ N \ \dots \ N)$ (where N is repeated $m+1$ times). From lemma 5.2.3, let $\mathcal{S} \subseteq [0, 1]$ be the full measure set of numbers with continued fraction expansions in which every finite sequence of positive integers occurs infinitely often. In particular, for $\omega \in \mathcal{S}$ the sequence S_N^{m+1} occurs in the continued fraction expansion of ω . Define $n_m(\omega)$ so that $a_{n_m+1} = \dots = a_{n_m+m+1} = N$.

Equation 5.4 shows that if $n = n_m + k$ with $0 \leq k < m$ then $\alpha_{n+1}/\alpha_n \leq (N+2)/N^3$.

Let $n = n_m + k$. Consider the trace of W_{p_n/q_n} at $f(\omega)$. Writing $\tau = q_n^{-2}(\omega - p_n/q_n)^{-1}$, we get that

$$\text{Tr } W_{p_n/q_n} = 2 - \frac{\pi^2}{\tau^2} + O(\tau^{-3}). \quad (5.6)$$

Applying conjecture 5.3.1, we get that

$$|f(p_n/q_n) - f(\omega)| = \Theta \left(\frac{\pi^2}{\tau^2 |\text{Tr}'_n|} \right).$$

That is,

$$|f(p_n/q_n) - f(\omega)| = \Theta \left(\frac{\pi^2 q_n^4 |\omega - p_n/q_n|^2}{|\text{Tr}'_n|} \right).$$

Applying the same reasoning at $f(z_n)$ instead of $f(p_n/q_n)$, and comparing we get that

$$|f(z_n) - f(\omega)| = \Theta \left(\frac{\pi^2 q_n^4 |\omega - p_n/q_n|^2}{|\text{Tr}'_n|} \right).$$

From this we get that

$$\sigma_n = \Theta \left(\frac{\pi^2 q_n^4 \alpha_n}{|\text{Tr}'_n|} \right).$$

Therefore,

$$\frac{\sigma_{n+1}}{\sigma_n} = \Theta \left(\frac{q_{n+1}^4}{q_n^4} \cdot \frac{\alpha_{n+1}}{\alpha_n} \cdot \frac{|\text{Tr}'_n|}{|\text{Tr}'_{n+1}|} \right).$$

Now $q_{n+1} \leq (N+1)q_n$, $\alpha_{n+1}/\alpha_n \leq (N+1)/N^3$ and by conjecture 5.4.1, $|\text{Tr}'_n|/|\text{Tr}'_{n+1}| = \Theta(1/N^3)$. It follows that $\sigma_{n+1}/\sigma_n = O(1/N)$. The constants involved are universal, so we can choose N large enough to make $\sigma_{n+1}/\sigma_n \leq 1/2$. With this choice of N , applying the same reasoning for each $n_m \leq n < n_m + m$ we get that $\sigma_{n_m+m}/\sigma_{n_m} \leq 1/2^m$.

This completes the argument for conjecture 5.5.1, but there are a few more things worth saying about this argument. First of all, although we have used a lot of implicit constants and less than symbols in showing $\sigma_{n+1}/\sigma_n = O(1/N)$, in fact if you assumed that the bounds were all reasonably tight then you would get $\sigma_{n+1}/\sigma_n \approx \pi^2/N$ for large enough N . This would suggest that you need to take $N = 10$ for the argument to work (that is, for $\sigma_{n+1}/\sigma_n \leq K < 1$ so that $\sigma_{n+m}/\sigma_n \rightarrow 0$ uniformly in m). Sure enough, running calculations on a computer show that $N = 9$ doesn't work and $N = 10$ does work, and that $\sigma_{n+1}/\sigma_n \approx \pi^2/N$ is reasonably accurate except for the first few N . For various computations of this sort, see the Mathematica notebook `sigma-scale-ratios.nb`.

Given any two sequences $A = (a_1 \dots a_n)$ and $B = (b_1 \dots b_k)$, define the sequence S_m to be A followed by m copies of B and p_m/q_m to be the corresponding fractions. Writing σ_m for the corresponding ratio, we get that σ_{m+1}/σ_m seems to depend almost entirely on the sequence B and not on A at all, and is approximately constant. In particular, for some sequences B this ratio is always greater

than 1, and for some sequences it is always less than 1. The only obvious exception is the sequence $B = (1)$ where the ratio is sometimes less than and sometimes greater than 1. Since the subsequence consisting of m copies of B appears in the continued fraction expansion of almost every ω , it appears likely that the quantity $\sigma_n(\omega)$ varies wildly between 0 and ∞ for almost all ω (although this isn't certain, it could tend to either 0 or ∞ almost everywhere).

It is also worth saying something about the sequence $B = (1)$. It has been conjectured that the point on the boundary of the Maskit slice corresponding to the golden mean $\phi = (\sqrt{5} - 1)/2$ (whose continued fraction is $[0 \ 1 \ 1 \ 1 \ \dots]$) is the lowest point. If this conjecture were true (and extensive computation suggests it is), it would rule out the possibility of the Maskit slice spiralling at that point (because if the boundary spiralled around it, there would have to be a point lower than it). In fact, it seems plausible that the boundary is conformal at this point (and every irrational point whose continued fraction has a tail of 1s only). Indeed, if p_n/q_n are the partial approximants to ϕ then computer calculations suggest that the initial direction of the pleating ray at the $2n$ th cusp is exactly vertical (and at the $2n + 1$ th cusp, approximately vertical). See the Mathematica notebook `golden-mean-sequence.nb`. This suggests that the irrational pleating ray at ϕ might extend beyond the boundary, which would show that the boundary was conformal at that point. For other continued fractions with a tail of 1s, it seems as though the initial directions of the pleating rays at the cusps corresponding to the partial approximants tend towards a fixed angle, or at least vary within a very narrow band, which suggests that the same may be true of these points too. This would be consistent with the ratio σ_{n+1}/σ_n varying around 1.

Finally, it is possible to relate the sequence B to the spiralling behaviour at a point. Whenever the sequence $(1 \ N \ 1 \ N \ \dots \ 1 \ N)$ of length $2m$ appears in the continued fraction expansion of ω , then at the scale of the cusps corresponding to the partial approximants in this subsequence, the boundary spirals around approximately $m/4$ times. Each appearance of $(1 \ N)$ introduces a rotation of angle approximately $\pi/2$ in the initial direction of the pleating rays. The sequence $(N \ N \ \dots \ N)$ doesn't correspond to any spiralling, because the first N introduces a rotation of $\pi/2$, the second one $-\pi/2$ and so forth. In particular, note that if the tail of the continued fraction expansion of ω consisted entirely of N 's (or indeed of any large integers), the boundary would not be expected to spiral infinitely at this point, but it wouldn't be conformal either.

5.6 Hausdorff dimension

The conjecture concerning the statistical distribution of sizes of $|\text{Tr}'_{p/q}(\mu_{p/q})|$ suggests an argument to show that the Hausdorff dimension of the boundary must be less than 1.25.

Let $U_{p/q}$ be the ball of radius $c_{p/q}$ about $\mu_{p/q}$, where

$$c_{p/q} := \frac{2}{|\text{Tr}'_{p/q}(\mu_{p/q})|}.$$

Numerical analysis (in Mathematica notebook `hausdorff.nb`) suggests strongly the following conjecture.

Conjecture 5.6.1 *For all N ,*

$$\bigcup_{\substack{p/q \\ q \geq N}} U_{p/q} \cap \mathcal{M} = \mathcal{M} - \{\mu_{p/q} : q < N\}.$$

That is, for any N , the class $\{U_{p/q} : q \geq N\}$ is a cover of $\mathcal{M}$ missing only finitely many points.

The cover is illustrated in figure 5.5. This conjecture is related to conjecture 5.3.1. If $U_{p/q}$ covers the image of the interval $I_{p/q} = (p/q - 1/q^2, p/q + 1/q^2)$, then the identity in conjecture 5.6.1 follows from the corresponding identity

$$\bigcup_{\substack{p/q \\ q \geq N}} I_{p/q} = [0, 1] - \{p/q : q < N\}.$$

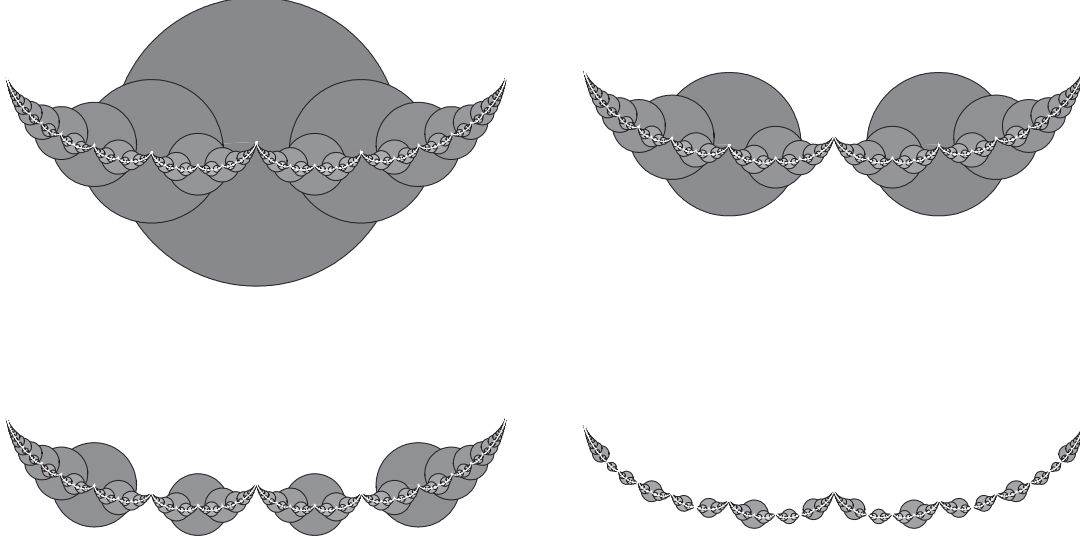


Figure 5.5: Covers of the Maskit slice by open balls $U_{p/q}$ with $q \geq 2$, $q \geq 3$, $q \geq 4$ and $q \geq 8$

This identity follows directly from lemma 5.2.4. Conjecture 5.3.1 implies that the image of the interval $I_{p/q}^\epsilon = (p/q - \epsilon/q^2, p/q + \epsilon/q^2)$ has diameter of order $\epsilon/|\text{Tr}'_{p/q}|$. Although it is true that the union of the intervals $I_{p/q}^\epsilon$ for all p/q with $q \geq N$ has full measure (by theorem 5.2.6), nonetheless this set could be compressed by the Riemann map into a set of measure 0 and so conjecture 5.3.1 is not sufficient.

Conjectures 5.4.3 and 5.6.1 together imply the following conjecture.

Conjecture 5.6.2 *The Hausdorff dimension of the Maskit slice satisfies $\dim_{\text{H}}(\mathcal{M}) < 1.25$.*

The Hausdorff d -dimensional measure $\Lambda_d(\mathcal{M})$ of the boundary is defined by

$$\Lambda_d(\mathcal{M}) = \lim_{\epsilon \rightarrow 0} \inf_{\substack{\text{covers } \{U_\alpha\} \\ \text{with} \\ \text{diam } U_\alpha \leq \epsilon}} \sum_{\alpha} (\text{diam } U_\alpha)^d.$$

In particular, if we define

$$\epsilon_N = \sup_{\substack{p/q \\ q \geq N}} \text{diam } U_{p/q} = \sup_{\substack{p/q \\ q \geq N}} c_{p/q}$$

then $\epsilon_N \rightarrow 0$ as $N \rightarrow \infty$ (by conjecture 5.4.3), and so

$$\Lambda_d(\mathcal{M}) \leq \lim_{N \rightarrow \infty} \sum_{\substack{p/q \\ q \geq N}} c_{p/q}^d.$$

Conjecture 5.4.3 tells us that for all p/q we have that $c_{p/q} \leq 2/q^{1.6}$. In the definition of the Hausdorff measure, the constant factor 2 is irrelevant so for ease of notation we will say $c_{p/q} \leq 1/q^{1.6}$. Therefore, if $d \geq 2/1.6 + \epsilon$ then $c_{p/q}^d \leq 1/q^{2+\epsilon}$. Therefore, we have that for sufficiently large N ,

$$\Lambda_d(\mathcal{M}) \leq \sum_{\substack{p/q \\ q \geq N}} c_{p/q}^d \leq \sum_{\substack{p/q \\ q \geq N}} \frac{1}{q^{2+\epsilon}} \leq \sum_{q \geq N} \frac{\phi(q)}{q^{2+\epsilon}} \leq \sum_{q \geq N} \frac{1}{q^{1+\epsilon}} < \infty.$$

Here $\phi(q)$ is the Euler totient function giving the number of integers $0 \leq p < q$ coprime to q which clearly satisfies $\phi(q) \leq q$. On average, $\phi(q) \approx 6q/\pi^2 \approx 0.6q$ so we can't improve on this. It follows that $\dim_{\text{H}}(\mathcal{M}) \leq 2/1.6 = 1.25$.

Conjecture 5.4.2 suggests that on average, $c_{p/q} \leq 1/q^{1.69846}$ which suggests the bound $\dim_{\text{H}}(\mathcal{M}) \leq 1.17754$ is likely to be correct.

In fact, both these bounds seem likely to be large overestimates. Write $\mu_{q,n}$ for the cusp corresponding to the n th fraction in the q th Farey series. That is, the $\mu_{q,n}$ are the ordered $\mu_{r/s}$ for $s \leq q$. Define $S_q^d = \sum_n |\mu_{q,n} - \mu_{q,n+1}|^d$. Here we are guessing that $|\mu_{q,n} - \mu_{q,n+1}|$ is a good estimate for the diameter of the segment of $\mathcal{M}$ between $\mu_{q,n}$ and $\mu_{q,n+1}$ (which seems likely after inspecting the Maskit slice at various scales), and that $\lim_{q \rightarrow \infty} S_q^d$ is a good estimate of $\Lambda_d(\mathcal{M})$. Figure 5.6 shows a graph of S_q^d against q for various values of d . At $d = 1$, it seems clear that $S_q^d \rightarrow \infty$ as $q \rightarrow \infty$. At $d = 1.6$ it seems clear that $S_q^d \rightarrow 0$. The turning point between these two behaviours seems to be around $d = 1.0582$ to $d = 1.0584$. This suggests $\dim_{\text{H}}(\mathcal{M}) \approx 1.058$.

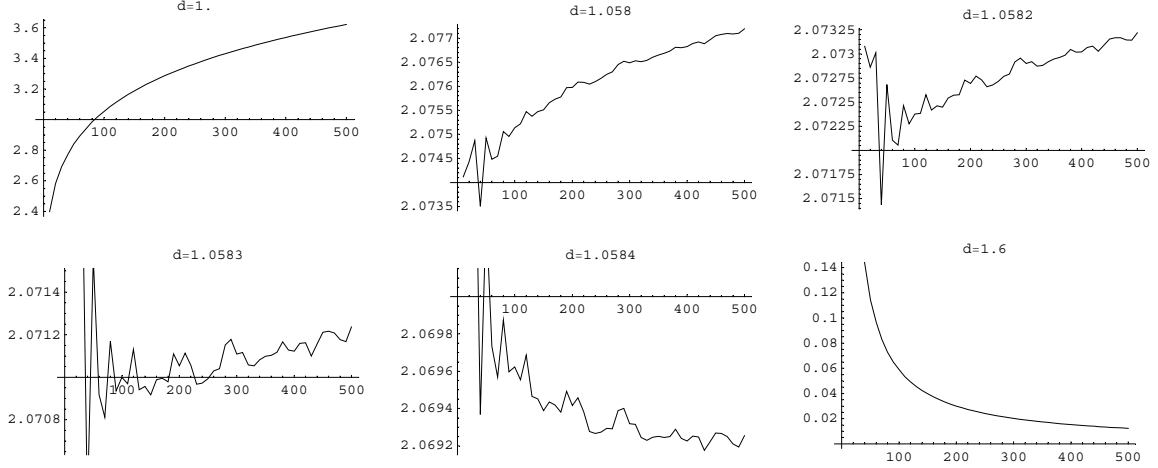


Figure 5.6: Graphs of estimated Hausdorff measure for various values of d

Chapter 6

Algorithms and programs

The purpose of this chapter is to describe in some detail the algorithms used to give evidence for the conjectures described earlier. Hopefully this might prove useful to others wanting to use numerical computation to suggest conjectures in this area.

6.1 Mathematica

6.1.1 Mathematica and C++

Throughout this chapter, the algorithms will be illustrated with Mathematica code because this is typically better known to mathematicians than C++ code and is usually clearer and conceptually easier to understand. However, Mathematica code is also quite slow for the sorts of calculations involved here, and so C++ was actually used for the long computations. The Mathematica notebook `algorithms-chapter.nb` includes all of the code in this chapter and some more.

6.1.2 A brief introduction to Mathematica

Function definitions

In Mathematica, lines of code are separated by semicolons. The symbol `:=` means that the left hand side is defined to be replaced by the right hand side. The square bracket symbols `[` and `]` are used for function definitions, and the `_` symbol is used in a function definition after a variable name. So for example `f[x_]:=x^2;` is the Mathematica definition of $f(x) = x^2$.

You can give recursive function definitions in Mathematica. The following is a recursive definition of the factorial function.

```
Fac[0]:=1;  
Fac[n_]:=n Fac[n-1];
```

Conditional functions and definitions

The Mathematica function `If[cond,a,b]` returns `a` if `cond` is true or `b` otherwise. Alternatively, conditionals can be expressed in the function definition by putting `;/cond` after the definition. For example, we could tighten up the definition of the factorial function as follows.

```
Fac[0]:=1  
Fac[n_]:=n Fac[n-1]/;n>0  
Fac[n_]:=Infinity;/;n<0
```

Modules

It is also possible to write more traditional programs in Mathematica using the `Module[{locals},program]` function. This evaluates the multi-line program `program` where `locals` is a comma separated list of local variables used only in that module. Values are returned from the module either by including the command `Return[value]` or by writing the return value as the last line of program without a semicolon at the end. For example, the factorial algorithm could be written more traditionally as follows.

```
Fac[n_] := Module[{ },
  If[n == 0, Return[1]];
  If[n < 0, Return[Infinity]];
  n Fac[n-1]
];
```

Loops

The function `For[init,cond,step,program]` is a standard programming for-loop. It first evaluates `init`, and then repeatedly evaluates `program` followed by `step` until `cond` is true. For example, we could rewrite the factorial program as follows.

```
Fac[n_] := Module[{result, i},
  result = 1;
  For[i = 2, i <= n, i++,
    result = result * i;
  ];
  result];
```

The statement `i++` is a shorthand for `i=i+1` which evaluates `i+1` and sets `i` to this new value (i.e. it just increases the value of `i` by 1).

Lists

A list in Mathematica is an ordered sequence of objects, which could be numbers, symbolic expressions or lists for example. Lists are written like sets with curly braces and commas, for example `{a,b,c}`. You can extract a particular element of a list using double square brackets. For example, if `squares={1,4,9,16}`, then `squares[[3]]` will evaluate to 9. There are a huge number of functions which operate on lists in Mathematica. Two of the more useful ones used below are `Join[list1,list2]` and `Append[list,element]`. The length of a list is given by `Length[list]`.

6.2 Computing the Boundary

6.2.1 Recursive formula for trace

The basic element in these calculations is the algorithm for computing the trace of the p/q -word. This recursive formula is due to David Wright, see [Wright88] and [MSW02]. We start from the formula for two matrices $M, N \in \text{SL}_2\mathbb{C}$,

$$\text{Tr } MN + \text{Tr } M^{-1}N = \text{Tr } M \cdot \text{Tr } N.$$

If $p/q < r/s$, so that $W_{(p+r)/(q+s)} = W_{p/q}W_{r/s}$ (see section 2.3.2), then

$$\text{Tr } W_{(p+r)/(q+s)} = \text{Tr } W_{p/q} \cdot \text{Tr } W_{r/s} - \text{Tr } W_{p/q}^{-1}W_{r/s}.$$

It is relatively easy to show that $\text{Tr } W_{(r-p)/(s-q)} = \text{Tr } W_{p/q}^{-1}W_{r/s}$, and so

$$\text{Tr } W_{(p+r)/(q+s)} = \text{Tr } W_{p/q} \cdot \text{Tr } W_{r/s} - \text{Tr } W_{(r-p)/(s-q)}. \quad (6.1)$$

Consider the following table (ignore the fourth column for the moment).

p/q	r/s	$(r-p)/(s-q)$	move
0/1	1/1	1/0	
0/1	1/2	1/1	L
0/1	1/3	1/2	L
1/4	1/3	0/1	R
1/4	2/7	1/3	L
3/11	2/7	1/4	R
5/18	2/7	3/11	R

This table gives us a way to compute $\text{Tr } W_{5/18}$. For the Maskit slice, we know that $\text{Tr } W_{0/1} = -i\mu$, $\text{Tr } W_{1/1} = -i\mu + 2i$ and $\text{Tr } W_{1/0} = 2$. So we know the traces of the p/q -words for the first row of the table. Equation 6.1 tells us that we can compute $\text{Tr } W_{1/2}$ from this, and so we know the traces of the second row. Each row involves only one new fraction, so we can always use equation 6.1 to compute the next row given the previous row.

The fourth column of the table specifies whether you had to “move left” or “move right” to get to this row from the previous one. A move left means going from $(p/q, r/s, (r-p)/(s-q))$ to $(p/q, (p+r)/(q+s), r/s)$. A move right means going from $(p/q, r/s, (r-p)/(s-q))$ to $((p+r)/(q+s), r/s, p/q)$. If (x, y, z) are the corresponding traces, then a move left corresponds to $(x, xy - z, y)$ and a right move to $(xy - z, y, x)$. In either case, you divide the interval $[p/q, r/s]$ at the point $(p+r)/(q+s)$ and look at either the left or right half of this interval. We encode this in the following pair of maps $\hat{\mathbb{Q}}^3 \times \mathbb{C}^3 \rightarrow \hat{\mathbb{Q}}^3 \times \mathbb{C}^3$:

$$\begin{aligned} L : (p/q, r/s, \cdot, x, y, z) &\longmapsto (p/q, (p+r)/(q+s), r/s, x, xy - z, y) \\ R : (p/q, r/s, \cdot, x, y, z) &\longmapsto ((p+r)/(q+s), r/s, p/q, xy - z, y, x) \end{aligned}$$

For a suitable sequence of L, R moves, you can get to any fraction $p/q \in [0, 1]$ starting from $(0/1, 1/1, 1/0)$. See sections 2.3.1 and 2.3.2. This can be expressed algorithmically very simply. Suppose your target fraction is u/v and you have reached $(p/q, r/s, \cdot)$. You simply check if $(p+r)/(q+s)$ is greater than, less than or equal to u/v , and do a corresponding left move, right move or stop. The following Mathematica code implements this algorithm.

```
f1⊕f2:=(Numerator[f1]+Numerator[f2])/(Denominator[f1]+Denominator[f2]);
TrF[uv_,uv_,rs_,x_,y_,z_]:=x;
TrF[uv_,pq_,uv_,x_,y_,z_]:=y;
TrF[uv_,pq_,rs_,x_,y_,z_]:=TrF[uv,pq,pq⊕rs,x,x y-z,y]/;pq⊕rs>uv
TrF[uv_,pq_,rs_,x_,y_,z_]:=TrF[uv,pq⊕rs,rs,x y-z,y,x]/;pq⊕rs<=uv
TrF[uv_,mu_]:=TrF[uv,0/1,1/1,-I mu,-I mu+2 I,2];
```

The first line of this program defines the Farey addition operator $\oplus$ by $p/q \oplus r/s = (p+r)/(q+s)$. The next four lines define the function TrF to be the value of $\text{Tr } W_{u/v}$ assuming $\text{Tr } W_{p/q} = x$, $\text{Tr } W_{r/s} = y$ and $\text{Tr } W_{(r-p)/(s-q)} = z$. The first two of these are the cases when $p/q = u/v$ or $r/s = u/v$ in which case x or y is returned. The third and fourth of these lines are the interesting cases. If $p/q \oplus r/s > u/v$ then we have a left move which is defined in the third of these lines, otherwise we have a right move defined in the fourth. The final line of the program just defines a shorthand $\text{TrF}[uv, \mu]$ for $\text{Tr } W_{u/v}(\mu)$ in the Maskit slice.

The trace of the p/q -word in the Maskit slice is a polynomial of degree q in μ with integer coefficients. This suggests that the simplest way of computing the trace would be to compute this polynomial once and then use this to evaluate it for particular μ . This has two problems. First of all, it requires $O(q)$ memory storage, and it also requires $O(q)$ operations to evaluate. On the other hand, the recursive method above requires $O(1)$ storage, and $O(\log q)$ operations to evaluate. In practice, this makes a very significant difference. Mathematica, which by default works with the polynomial representation, struggles to do the computations below for $q = 20$, whereas the C++ program can easily handle $q = 20000$. The second problem is that if you evaluate the polynomial using finite precision arithmetic, you get numerical instability. Experimentally, this seems to be a problem for $q = 23$ and onwards on an ordinary PC. This is not a problem for the recursive method. Note that it

is possible to force Mathematica to use the recursive method, which makes it possible to do almost as much with Mathematica as with C++, but it still an interpreted language and is much slower than C++. See the Mathematica notebook `algorithms-chapter.nb` to see how to do this.

For reference, the more traditional way of writing this program, which would be more useful for writing a C++ implementation for example, is as follows.

```
FareyAdd[f1_,f2_] :=(Numerator[f1]+Numerator[f2])/(Denominator[f1]+Denominator[f2]);
TrF[uv_,pq_,rs_,x_,y_,z_] :=Module[{pqrs},
  If[uv==pq,Return[x]];
  If[uv==rs,Return[y]];
  pqrs=FareyAdd[pq,rs];
  If[pqrs>uv,
    TrF[uv,pq,pqrs,x,x y-z,y],
    TrF[uv,pqrs,rs,x y-z,y,x]]
];
```

As a final note on the implementation (for programmers). This recursive definition of the trace functions requires $O(\log q)$ function calls and therefore $O(\log q)$ storage space on the stack. Although this is not much of a problem, it is a fairly easy exercise to rewrite this function without using recursion. This is how David Wright's pseudo-code algorithm on page 286 of [MSW02] works. The author's C++ function `maskit::trace_poly` in `maskitalgorithms.cpp` is straight out of [MSW02] and also uses this non-recursive algorithm.

6.2.2 Finding cusps I

The basic idea behind drawing the boundary of the Maskit slice is to use the fact that cusps are dense in the boundary and are enumerated in the correct order by fractions p/q . We plot the boundary by plotting every p/q -cusp for $q \leq q_{\max}$. At the p/q -cusp, the trace of the p/q -word is ± 2 , as $W_{p/q}$ is parabolic. For the Maskit slice, this trace is always $+2$. We already have an efficient algorithm for computing $\text{Tr } W_{p/q}(\mu)$ from the previous section, so we need only find the correct root $\mu_{p/q}$ of $\text{Tr } W_{p/q}(\mu) = 2$. David Wright's solution was to start with $\mu_{0/1} = 2i$ which we know, and use this root as the initial guess in applying Newton's method to find the next cusp. This new cusp is then used as the initial guess to find the next cusp, and so forth. The hope is that the continuity of the boundary will mean that this initial guess forces Newton's method to converge to the correct root of the trace polynomial for the next cusp. In practice, for the Maskit slice and the Earle slice, this is true. For other slices, this may not work and we discuss an alternative algorithm below in these cases.

We start by giving an algorithm using built in functions of Mathematica to illustrate the process. The numerical stability issues mentioned in the previous section mean this method can only be used to compute the boundary cusps p/q with $q \leq 23$. The program below only runs for $q \leq 15$ (so that it runs quickly).

```
qmax=15;
FareyList=Union[Flatten[Table[p/q,{q,1,qmax},{p,0,q}]]];
initmu=2.I;
boundary={};
For[i=1,i<=Length[FareyList],i++,
  tracefunction=TrF[FareyList[[i]],mu];
  initmu=mu/.FindRoot[tracefunction==2,{mu,initmu}];
  boundary=Append[boundary,initmu];
];
```

This program requires some explanation. The `FareyList=...` line is just a little Mathematica trick to make the variable `FareyList` an ordered list of the fractions p/q with $q \leq q_{\max}$. The `initmu=2.I;` line just says that the first guess we use will be $\mu = 2i$ (which is in fact the correct solution for the $0/1$ -cusp). The `boundary={}` line just sets `boundary` to be an empty list. By the end, it will be a list of complex numbers corresponding to the p/q -cusps for the values of p/q in `FareyList`. The `For` line just evaluates the next three lines for each of the values of i from 1 to `Length[FareyList]`. The next line sets `tracefunction` to be the symbolic expression returned by

$\text{TrF}[\text{FareyList}[[i]], \mu]$, which is the trace of the p/q -word in μ where $p/q = \text{FareyList}[[i]]$, the i th element of the list of fractions. The function TrF is the one defined in the previous section. The expression $\mu/. \text{FindRoot}[\text{tracefunction}==2, \{\mu, \text{initmu}\}]$ in Mathematica just returns the result of applying Newton's algorithm to the function $\text{tracefunction}-2$ starting with an initial guess initmu . In fact, it uses a more sophisticated algorithm than Newton's algorithm, but the effect is the same (see the Mathematica documentation for more details). This value will be the new initial guess for the next cusp. Finally, the list `boundary` has this new cusp appended to it.

To write a C++ implementation (or any other language which doesn't have Mathematica's built in functions), we need to write programs for Newton's method, for finding the derivative of the function, and also for listing the fractions p/q with $q \leq \text{qmax}$ in order. Wright has written an excellent guide to writing these functions in [MSW02] pages 286–309. Alternatively, you could study the C++ implementation used here. Newton's method is implemented in the file `newton.hpp`, function `tsolve_analytic`. The function giving the next fraction in the Farey sequence is given in file `farey.cpp` function `next_in_sequence`. The function which puts this all together is in file `maskitalgorithms.cpp` function `view::compute_boundary`.

We focus here on one aspect of Wright's algorithm, the computation of the derivative of the trace function. Newton's algorithm for finding roots $f(z) = 0$ proceeds by starting with an initial guess z_0 and repeatedly applying the rule

$$z \mapsto z - \frac{f(z)}{f'(z)}$$

to get a sequence $z_n \rightarrow z_\infty$ such that $f(z_\infty) = 0$. To use this method in our case, we need to be able to compute $\text{Tr}'_{p/q}$. Wright's solution is to use a numerical approximation of the derivative. We could just choose a small value ϵ and compute

$$\frac{f(z + \epsilon) - f(z)}{\epsilon},$$

but in fact Wright uses the more accurate approximation (for analytic functions)

$$\frac{f(z + \epsilon) - f(z - \epsilon)}{4\epsilon} + \frac{f(z + i\epsilon) - f(z - i\epsilon)}{4i\epsilon}.$$

This approximation is accurate to order ϵ^4 . This is the algorithm used in the C++ implementation used throughout. However, there is an alternative, marginally quicker and even more accurate way of computing the trace derivative.

Recall the L and R moves of the previous section. Under an L move, the fractions $(p/q, r/s, (r - p)/(s - q))$ are replaced by $(p/q, (p + r)/(s + q), r/s)$ and the triple of corresponding traces (x, y, z) is replaced by $(x, xy - z, y)$. Suppose we had already computed the derivatives (with respect to μ) of these traces to be $(\dot{x}, \dot{y}, \dot{z})$. The derivative of the trace function corresponding to $(p + r)/(q + s)$ will then be $\dot{x}y + x\dot{y} - \dot{z}$. So the triple of trace derivatives can be replaced by $(\dot{x}, \dot{x}y + x\dot{y} - \dot{z}, \dot{y})$. A similar procedure works for an R move. The following Mathematica code uses this to evaluate the pair $(\text{Tr}_{u/v}(\mu), \text{Tr}'_{u/v}(\mu))$.

```
FullTrF[uv_, uv_, rs_, x_, y_, z_, dx_, dy_, dz_] := x;
FullTrF[uv_, pq_, uv_, x_, y_, z_, dx_, dy_, dz_] := y;
FullTrF[uv_, pq_, rs_, x_, y_, z_, dx_, dy_, dz_] :=
  FullTrF[uv, pq, pq⊕rs, x, x y - z, y, dx, x dy + dx y - dz, dy] /; pq⊕rs > uv
FullTrF[uv_, pq_, rs_, x_, y_, z_, dx_, dy_, dz_] :=
  FullTrF[uv, pq⊕rs, rs, x y - z, y, x, x dy + dx y - dz, dy, dx] /; pq⊕rs <= uv
FullTrF[uv_, mu_] := FullTrF[uv, 0/1, 1/1, -I mu, -I mu + 2 I, 2, -I, -I, 0];
```

6.2.3 Pleating rays

It is fairly easy to compute the pleating rays using the algorithms above. The p/q -pleating ray $\wp_{p/q}$ in the Maskit slice is the unique branch of $\text{Tr}_{p/q}(\mu) \in [2, \infty)$ which touches the cusp $\mu_{p/q}$. So, the algorithm for computing it is simple: start with $\wp_{p/q}(t) = \mu_{p/q}$ corresponding to $t = 2$, and slowly increase t using Newton's method to find the corresponding root of $\text{Tr}_{p/q}(\mu) = t$ at each stage.

Both the Mathematica and C++ routines do this. The only subtlety is in the choice of which finite

subset $T \subseteq [2, \infty)$ of trace values t to use. This is not a mathematical issue though, purely one of presentation. You want to compute the least number of points which make it look right. The C++ routines use an automatically adapting technique so that the points in the μ plane are approximately evenly spaced. The Mathematica routines use a simpler technique to make it conceptually simpler. See the files for more information.

6.2.4 Finding cusps II

The Maskit slice has some very nice features which make it amenable to computation. It is possible that the algorithm used to draw the boundary wouldn't work for other slices because using a close cusp as the initial guess for Newton's method would find the wrong solution of the trace polynomial. There is an alternative algorithm using pleating rays.

For the Maskit slice, Linda Keen and Caroline Series proved in [KeenSeries93] that the p/q -pleating ray $\wp_{p/q}$ is asymptotic to the line $\text{Re}(\mu) = 2p/q$ as $\text{Im}(\mu) \rightarrow +\infty$. We start by choosing a point $2p/q + iy$ for some large value of y . This will be close to $\wp_{p/q}$. Using this as an initial guess, we find a point μ on the pleating ray. We then compute the value $t = \text{Tr}_{p/q} \mu$. We now work our way down the pleating ray until we get to $t = 2$ which will correspond to the cusp $\mu_{p/q}$ using Newton's method at each stage.

This suggests a general algorithm for other slices. The details will vary, but the general scheme is as follows.

1. Enumerate the cusps combinatorially. For the Maskit, Earle and Bers slices, this can be done using fractions p/q .
2. Find a recursive formula for the trace functions associated to cusps.
3. Find the asymptotic behaviour of the pleating rays (or possibly pleating planes for higher dimensional cases).
4. For each cusp, choose an asymptotic point on the pleating ray and work down towards the cusp using Newton's method.

This general algorithm has already been used to plot the pleating rays in the Earle and Bers slices. For the Earle slice, it is proved in [KomSer01] that $\wp_{p/q}$ intersects the real axis at the point $b_{p/q}$ which is the unique critical point of $\text{Tr}_{p/q}(d)$ on $\mathbb{R}^+$. Note that from the definitions in section 3.2.3 it is clear that $\text{Tr}_{p/q}(d) \in \mathbb{R}$ when $d \in \mathbb{R}$. So to find $\wp_{p/q}$ you start by finding the unique critical point $b_{p/q}$ on $\mathbb{R}^+$ (which is simple and quick, even the most basic subdivision algorithm is pretty efficient for doing this).

For the Bers slice, it is proved in [KomSug04] that if $\mathcal{H}_{p/q}$ is the locus of points $\phi \in \mathcal{B}$ such that $\text{Tr}_{p/q}^2(\phi) \in (4, \infty)$ then $\wp_{p/q}$ is the unique connected component of $\mathcal{H}_{p/q} - \{0\}$ on which $\text{Tr}_{p/q}^2(\phi) < \text{Tr}_{p/q}^2(0)$. Komori and Sugawa use this fact to plot the pleating rays by starting at 0 and moving down the pleating ray.

For the Maskit slice, the following Mathematica code illustrates the algorithm described above.

```
FindCuspPleating[f_] := Module[{t, dt, t0, m0},
  t[mu_] := TrF[f, mu];
  dt[mu_] := DTrF[f, mu];
  m0 = 2.f + 10.I;
  t0 = Re[t[m0]];
  While[Abs[t[m0] - 2.] > 0.0001,
    m0 = m0 - (t[m0] - t0)/dt[m0];
    t0 = 2 + .75(t0 - 2);
  ];
  m0
];
```

The initial guess is the point $\mu_0 = 2p/q + 10i$. Experimentally, this choice seems to work for the Maskit slice, whereas the choice $2p/q + 3i$ sometimes leads to the algorithm finding the wrong root

(see figure 3.1 for an idea as to why this might happen). Rather than tracing the pleating ray directly, this algorithm first guesses that $t_0 = \text{Re}(\text{Tr}_{p/q}(\mu_0))$ is the trace of a point on $\mathcal{S}_{p/q}$ near to μ_0 . The algorithm then applies one step of Newton's method, and replaces t_0 by $2 + (3/4)(t_0 - 2)$. Another step of Newton's method is then applied, and so on until the trace of μ_0 is within 0.0001 of 2. Essentially this is Newton's method applied to a moving target. This suffices for this algorithm because we are not interested in finding the pleating ray per se, only in finding the endpoint.

Appendix A

Notation

We have used the following notation. The bullet point lists give variations on a general notation.

$\mathbb{H}$	The hyperbolic upper half plane.
$\mathbb{H}^2$	
$\mathbb{H}^3$	The hyperbolic upper half space.
G	Usually a discrete subgroup of $\mathrm{PSL}_2\mathbb{C}$.
Γ	The group $\pi_1(\Sigma)$ or equivalently the free group on two generators $\langle X, Y \rangle$.
$\Omega^\pm$	The domains of discontinuity of a once-punctured torus group.
Λ	The limit set of G .
ρ	A representation $\Gamma \rightarrow \mathrm{PSL}_2\mathbb{C}$.
M	Usually the 3-manifold $\mathbb{H}^3 \cup \Omega^+ \cup \Omega^- / G$.
$\nu^\pm$	The Teichmüller parameters of $\Omega^\pm / G$.
Σ	A fixed once-punctured torus.
W	A word in the group $\Gamma = \langle X, Y \rangle$ or $\langle A, B \rangle$. • $W_{p/q}$ the p/q-word in Γ
μ	Coordinate for the Maskit slice. • μ_W the cusp corresponding to the word W • $\mu_{p/q}$ the p/q-cusp • μ_n, μ_∞ a sequence of cusps μ_n, all neighbours of μ_∞, tending to μ_∞ • $\tilde{\mu}_n$ an approximate cusp sequence
Tr	Trace of a matrix. • Tr_W function from a representation space or slice, with value $\mathrm{Tr}(\rho(W))$ where ρ is an element of the representation space or slice • $\mathrm{Tr}_{p/q}$ trace function for the p/q-word • $\mathrm{Tr}_n, \mathrm{Tr}_\infty$ sequence of trace functions associated to a sequence of cusps $\mu_n \rightarrow \mu_\infty$ • Tr' derivative of the trace function. For example, for the Maskit slice it is the derivative with respect to the coordinate μ

$\wp$	Pleating ray. • $\wp_W$ the pleating ray associated to the word W • $\wp_{p/q}$ the pleating ray ending at the p/q-cusp • $\wp(t)$ the function from $[2, \infty)$ to a slice or representation space parameterising the pleating ray so that $\text{Tr } \wp(t) = t$
$O,$ $\Omega,$ Θ	Complexity notation. A function $f(x)$ is • $O(g(x))$ if $\lim_{x \rightarrow \infty} f(x)/g(x) < \infty$. • $\Omega(g(x))$ if $\lim_{x \rightarrow \infty} f(x)/g(x) > 0$. • $\Theta(g(x))$ if $0 < \lim_{x \rightarrow \infty} f(x)/g(x) < \infty$.
$\hat{\mathbb{Q}}$	The set $\mathbb{Q} \cup \{\infty\}$. The element ∞ is written $1/0$.
$\hat{\mathbb{R}}$	The set $\mathbb{R} \cup \{\infty\}$.
$\oplus$	Farey addition, see section 2.3.1.

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